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On High Order MIRK Schemes and Hermite-Birkhoff Interpolants

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Abstract: Mono-Implicit Runge-Kutta (MIRK) formulae present an effective means for solving general non-linear two-point boundary value problems. High order finite difference schemes provide significant savings in both computational time and memory when the problem exhibits the required smoothness. In this paper we introduce MIRK methods of orders 10 & 12. The local truncation error of the tenth order scheme is presented and verified by numerical experiments. A piecewise Hermite-Birkhoff interpolant of order 10 is also introduced, which allows for *event locations* (such as roots or extrema) of the solution to be found. The corresponding error analysis and numerical data are provided for the interpolant as well. We perform numerical experiments on the set of 32 test problems from Cash & Wright [9], and find that the order 12 scheme provides a significantly greater accuracy than observed with the lower order schemes.

Keywords: MIRK, Runge Kutta, ODE, Boundary Value Problem, Hermite-Birkhoff Interpolation, Lobatto points

Mathematics Subject Classification: 65D05, 65L10, 65L12

1 Introduction

Many non-linear two point boundary value problems (BVPs) can be re-arranged in terms of the general first order system:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,\ y), \qquad a \le x \le b, \qquad g(y(a),\ y(b)) = 0. \tag{1.1}$$

We consider Mono-Implicit Runge Kutta (MIRK) formulae as a means for solving (1.1). In particular we concentrate our attention on a special class of symmetric MIRK formulae, which consist

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of a $2m+1$ point integration formula based on Lobatto points:

$$\begin{aligned}\frac{y_{n+1}-y_n}{h_n} =&\beta_0\left[f\left(x_{n+1},\ y_{n+1}\right)+f\left(x_n,\ y_n\right)\right]\\ &+\sum_{i=1}^{m-1}\beta_i\left[f\left(x_{n+\frac{1}{2}+\alpha_i},\ y_{n+\frac{1}{2}+\alpha_i}\right)+f\left(x_{n+\frac{1}{2}-\alpha_i},\ y_{n+\frac{1}{2}-\alpha_i}\right)\right]\\ &+\beta_m f\left(x_{n+\frac{1}{2}},\ y_{n+\frac{1}{2}}\right)\end{aligned} \tag{1.2a}$$

where the internal points $y_{n+\frac{1}{2}\pm\alpha_i}$ are computed as follows:

$$\begin{aligned}y_{n+\frac{1}{2}\pm\alpha_i} =&A_i^{\pm}y_{n+1}+A_i^{\mp}y_n\pm h_n\left\{B_i^{\pm}f\left(x_{n+1},\ y_{n+1}\right)+B_i^{\mp}f\left(x_n,\ y_n\right)\right.\\ &\left.+\sum_{j=1}^{N(i)<i}\left[C_{ij}^{\pm}f\left(x_{n+\frac{1}{2}+\alpha_j},\ y_{n+\frac{1}{2}+\alpha_j}\right)+C_{ij}^{\mp}f\left(x_{n+\frac{1}{2}-\alpha_j},\ y_{n+\frac{1}{2}-\alpha_j}\right)\right]\right\}\end{aligned} \tag{1.2b}$$

Finite difference schemes, such as (1.2), are usually solved numerically using a modified Newton method. As MIRK schemes are implicit only in the two variables y_n and y_{n+1} when solving BVPs; the resultant Newton Iteration Matrix (NIM) adopts a compact readily solvable structure. This allows for the rapid solution of the non-linear system and thus our problem (1.1). As the order of the MIRK scheme increases, however, the analytic complexity of the Newton Iteration Matrix (NIM) blocks also increases. This presents a problem when working with high order schemes as the final analytic expression for the NIM can become extremely complicated rendering it both expensive to evaluate, and potentially inaccurate. Most program codes (e.g. NRK or TWPBVP) can compute the NIM analytically for order 4 methods (using a pre-supplied Jacobian). For orders 6 and above, however, either the NIM is computed by numerical finite differences or deferred correction is carried out with an order 4 MIRK scheme that still has the tractable analytic form for the Jacobian. For reference, MIRK schemes of orders 4, 6 & 8 can be found in [5, 7, 3]. Our MIRK 10 & 12 schemes are implemented in Fortran 90 [2] and Maple using both a numerical NIM and as part of a deferred correction framework.

Solutions to (1.1) by MIRK methods consist of a discrete set of points rather than as a continuous solution. Cash and Moore [6] present an interpolation scheme for MIRK 6 & 8 methods where the internal MIRK points $y_{n+\frac{1}{2}\pm\alpha_i}$ are used along with the first derivative information $f\left(x_{n+\frac{1}{2}\pm\alpha_i},\ y_{n+\frac{1}{2}\pm\alpha_i}\right)$ to formulate piecewise Hermite-Birkhoff interpolants through the intervals $[x_n,\ x_{n+1}]$. In this paper we construct a Hermite-Birkhoff interpolant for our MIRK 10 scheme in a similar fashion and provide the corresponding error terms. Computing the interpolant is not straightforward, however, as the internal MIRK points do not posses a sufficient level of accuracy to form an order 10 interpolant. This issue is resolved by the computation of higher order points using the internal MIRK data and lower order interpolants; a process known as *bootstrapping*, a discussion of which can be found in [6]. Bootstrapping was used to derive the MIRK 10 & 12 schemes presented in this paper in a manner similar to that described in [11].

Throughout this paper we have performed various calculations using Maple. This has allowed us to carry out complicated algebraic manipulation rapidly. Wherever possible, however, this has been double checked using alternative software. For example, the coefficients of the MIRK 10 & 12 schemes have been computed using Maple, but the orders of the schemes have been verified numerically using our Fortran 90 code [2].

This paper is organised as follows: In section 2, we introduce the MIRK 10 finite difference scheme, and provide detailed truncation error information. MIRK 12 is also briefly introduced, a Fortran 90 implementation of MIRK 12 can be found in [2]. We discuss the order 10 accurate Hermite-Birkhoff interpolant in section 3. A comparison between Orszag's method [12] and our MIRK 10 & 12 schemes for solving the Orr-Sommerfeld equation can be found in section 4. More comprehensive numerical testing is performed in section 5. Concluding remarks and suggested areas for further research can be found in section 6.

2 The MIRK 10 & 12 Finite Difference Schemes

We use the superscript notation $y^{(a)}_{n+\frac{1}{2}+\omega}$ to denote a point at $x = x_n + h_n(\frac{1}{2} + \omega)$ with final $O(h_n^a)$ truncation error (we take account of the truncation errors of the intermediate points). For $\alpha_5 = \frac{\sqrt{495-66\sqrt{15}}}{66}$ and $\alpha_6 = \frac{\sqrt{495+66\sqrt{15}}}{66}$, the following formula based on Lobatto points has local truncation error $O(h_n^{12})$:

$$\begin{aligned} y_{n+1} - y_n = \frac{h_n}{2100}\Big[&\left(372 + 21\sqrt{15}\right)\left(y'^{(8)}_{n+\frac{1}{2}+\alpha_5} + y'^{(8)}_{n+\frac{1}{2}-\alpha_5}\right) \\ &+ \left(372 - 21\sqrt{15}\right)\left(y'^{(8)}_{n+\frac{1}{2}+\alpha_6} + y'^{(8)}_{n+\frac{1}{2}-\alpha_6}\right) \\ &+ 512 y'^{(8)}_{n+\frac{1}{2}} + 50\left(y'_{n+1} + y'_n\right)\Big] \end{aligned} \tag{2.1}$$

where the $y'^{(8)}_{n+\frac{1}{2}\pm\alpha_{5,6}}$ and $y'^{(8)}_{n+\frac{1}{2}}$ are calculated using the bootstrapping process below. The relevant constants, $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, r_i, \alpha_i\}$, can be found in appendix C where we note in particular the important fact that all α_i are in the range $(0, \frac{1}{2})$. By computing derivatives using the problem formulation (1.1):

$$y'^{(a)}_{n+\frac{1}{2}\pm\alpha} = f\left(x_{n+\frac{1}{2}\pm\alpha}, y^{(a)}_{n+\frac{1}{2}\pm\alpha}\right) \tag{2.2}$$

we are able to compute the following points.

Order 4 points:

$$y^{(4)}_{n+\frac{1}{2}\pm\alpha_1} = a_1^{\pm} y_n + a_1^{\mp} y_{n+1} \mp h_n\left(a_2^{\pm} y'_n + a_2^{\mp} y'_{n+1}\right) \tag{2.3}$$

Order 5 points:

$$y^{(5)}_{n+\frac{1}{2}\pm\alpha_2} = b_1^{\pm} y_n + b_1^{\mp} y_{n+1} \mp h_n\left(b_2^{\pm} y'_n + b_2^{\mp} y'_{n+1} + b_3^{\pm} y'^{(4)}_{n+\frac{1}{2}-\alpha_1} + b_3^{\mp} y'^{(4)}_{n+\frac{1}{2}+\alpha_1}\right) \tag{2.4}$$

Order 6 points:

$$y^{(6)}_{n+\frac{1}{2}\pm\alpha_3} = c_1^{\pm} y_n + c_1^{\mp} y_{n+1} \mp h_n\left(c_2^{\pm} y'_n + c_2^{\mp} y'_{n+1} + c_3^{\pm} y'^{(5)}_{n+\frac{1}{2}-\alpha_2} + c_3^{\mp} y'^{(5)}_{n+\frac{1}{2}+\alpha_2}\right) \tag{2.5}$$

Order 7 off-centre points:

$$y^{(7)}_{n+\frac{1}{2}\pm\alpha_4} = d_1^{\pm} y_n + d_1^{\mp} y_{n+1} \mp h_n\left(d_2^{\pm} y'_n + d_2^{\mp} y'_{n+1} + d_3^{\pm} y'^{(6)}_{n+\frac{1}{2}-\alpha_3} + d_3^{\mp} y'^{(6)}_{n+\frac{1}{2}+\alpha_3}\right) \tag{2.6}$$

Order 7 midpoint:

$$\begin{aligned} y^{(7)}_{n+\frac{1}{2}} = \frac{1}{2}\left(y_n + y_{n+1}\right) + h_n\Big(&e_1\left[y'_n - y'_{n+1}\right] + e_2\left[y'^{(6)}_{n+\frac{1}{2}-\alpha_3} - y'^{(6)}_{n+\frac{1}{2}+\alpha_3}\right] \\ &+ e_3\left[y'^{(7)}_{n+\frac{1}{2}-\alpha_4} - y'^{(7)}_{n+\frac{1}{2}+\alpha_4}\right]\Big) \end{aligned} \tag{2.7}$$

Order 8 off-centre points:

$$y^{(8)}_{n+\frac{1}{2}\pm\alpha_5} = f_1^{\pm} y_n + f_1^{\mp} y_{n+1} \mp h_n \Big(f_2^{\pm} y'_n + f_2^{\mp} y'_{n+1} \pm f_3 \left[y'^{(6)}_{n+\frac{1}{2}+\alpha_3} - y'^{(6)}_{n+\frac{1}{2}-\alpha_3} \right] + f_4^{\pm} y'^{(7)}_{n+\frac{1}{2}-\alpha_4} + f_4^{\mp} y'^{(7)}_{n+\frac{1}{2}+\alpha_4} + f_5 y'^{(7)}_{n+\frac{1}{2}} \Big) \tag{2.8}$$

$$y^{(8)}_{n+\frac{1}{2}\pm\alpha_6} = g_1^{\pm} y_n + g_1^{\mp} y_{n+1} \mp h_n \Big(g_2^{\pm} y'_n + g_2^{\mp} y'_{n+1} \pm g_3 \left[y'^{(6)}_{n+\frac{1}{2}+\alpha_3} - y'^{(6)}_{n+\frac{1}{2}-\alpha_3} \right] + g_4^{\pm} y'^{(7)}_{n+\frac{1}{2}-\alpha_4} + g_4^{\mp} y'^{(7)}_{n+\frac{1}{2}+\alpha_4} + g_5 y'^{(7)}_{n+\frac{1}{2}} \Big) \tag{2.9}$$

Order 8 midpoint:

$$y^{(8)}_{n+\frac{1}{2}} = \frac{1}{2}(y_n + y_{n+1}) + h_n \Big(r_1 \left[y'_n - y'_{n+1} \right] + r_2 \left[y'^{(7)}_{n+\frac{1}{2}-\alpha_4} - y'^{(7)}_{n+\frac{1}{2}+\alpha_4} \right] + r_3 \left[y'^{(8)}_{n+\frac{1}{2}-\alpha_5} - y'^{(8)}_{n+\frac{1}{2}+\alpha_5} \right] + r_4 \left[y'^{(8)}_{n+\frac{1}{2}-\alpha_6} - y'^{(8)}_{n+\frac{1}{2}+\alpha_6} \right] \Big) \tag{2.10}$$

With the exception of $y^{(8)}_{n+\frac{1}{2}}$, all the $y^{(j)}_{n+\frac{1}{2}\pm\alpha_i}$ are computed from degree $j-1$ Hermite-Birkhoff interpolants, examples of which can be found in section 3. A degree 9 interpolant is actually used to compute the eighth order accurate point $y^{(8)}_{n+\frac{1}{2}}$ rather than a degree 7 one. This gives us two new variables p and q such that:

$$y^{(8)}_{n+\frac{1}{2}} - y\left(x_n + \frac{h_n}{2}\right) = p h_n^8 + q h_n^9 + O(h_n^{10}). \tag{2.11}$$

The variables p and q provide us with 2 extra degrees of freedom which allow us to cancel out the $O(h_n^{10})$ error terms of the MIRK 10 scheme as we now explain. We can visualise the bootstrapping process as follows:

$$\begin{array}{cccccc} & y^{(4)}_{n+\frac{1}{2}-\alpha_1} & & y^{(4)}_{n+\frac{1}{2}+\alpha_1} & & O(h_n^4) \\ & y^{(5)}_{n+\frac{1}{2}-\alpha_2} & & y^{(5)}_{n+\frac{1}{2}+\alpha_2} & & O(h_n^5) \\ & y^{(6)}_{n+\frac{1}{2}-\alpha_3} & & y^{(6)}_{n+\frac{1}{2}+\alpha_3} & & O(h_n^6) \\ & \Updownarrow & & \Updownarrow & & \alpha_3 = \frac{1}{6}\sqrt{24\alpha_4^2+3} \\ & y^{(7)}_{n+\frac{1}{2}-\alpha_4} & y^{(7)}_{n+\frac{1}{2}} & y^{(7)}_{n+\frac{1}{2}+\alpha_4} & & O(h_n^7) \\ y^{(8)}_{n+\frac{1}{2}-\alpha_5} & y^{(8)}_{n+\frac{1}{2}+\alpha_5} & y^{(8)}_{n+\frac{1}{2}} & y^{(8)}_{n+\frac{1}{2}-\alpha_6} & y^{(8)}_{n+\frac{1}{2}+\alpha_6} & O(h_n^8) \end{array} \tag{2.12}$$

In order to derive the $O(h_n^7)$ accurate terms we need to impose the relation $\alpha_3 = \frac{1}{6}\sqrt{24\alpha_4^2+3}$. We can solve for α_3 and p to cancel out the $O(h_n^8)$ and $O(h_n^9)$ error terms of our MIRK scheme. As our MIRK scheme is symmetric, this ensures that the $O(h_n^{10})$ terms also cancel, leaving us with an $O(h_n^{11})$ finite difference scheme hence a 10^{th} order MIRK scheme.

In order to increase the accuracy of our method further we consider the error terms of (2.1) when dealing with problems of the following linear form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,\ y) = Ay + b \qquad \text{where } A \text{ and } b \text{ are constant} \tag{2.13}$$

By solving for α_2, α_3, p and q; we can cancel out the $O(h_n^{11})$ error terms of our MIRK scheme when dealing with problems of the form (2.13). Thus our scheme will be order 12 accurate for certain linear problems. This would not be possible if our integration formula was tenth order accurate, and indeed this is the principal reason we have elected to choose a formula of order 12 which is based on Lobatto points. The eighth order accurate scheme found in [7], increases to tenth order accuracy in a similar fashion when solving problems of the form (2.13). We arbitrarily choose $\alpha_1 = \frac{1}{4}$, to simplify the algebra, as we have found no other choice for α_1 which decreases the truncation error or zeroes a bootstrapping coefficient.

We have also derived a MIRK 12 method in a similar way, its bootstrapping scheme can be summarised as:

$$\begin{array}{cccccc}
 & y^{(4)}_{n+\frac{1}{2}-\beta_1} & & y^{(4)}_{n+\frac{1}{2}+\beta_1} & & O(h_n^4) \\
 & y^{(5)}_{n+\frac{1}{2}-\beta_2} & & y^{(5)}_{n+\frac{1}{2}+\beta_2} & & O(h_n^5) \\
 & y^{(6)}_{n+\frac{1}{2}-\beta_3} & & y^{(6)}_{n+\frac{1}{2}+\beta_3} & & O(h_n^6) \\
 & \Updownarrow & & \Updownarrow & & \beta_3 = \frac{1}{6}\sqrt{24\beta_4^2+3} \\
 & y^{(7)}_{n+\frac{1}{2}-\beta_4} & y^{(7)}_{n+\frac{1}{2}} & y^{(7)}_{n+\frac{1}{2}+\beta_4} & & O(h_n^7) \\
y^{(8)}_{n+\frac{1}{2}-\beta_5} & y^{(8)}_{n+\frac{1}{2}+\beta_5} & & y^{(8)}_{n+\frac{1}{2}-\beta_6} & y^{(8)}_{n+\frac{1}{2}+\beta_6} & O(h_n^8) \\
\Updownarrow & \Updownarrow & & \Updownarrow & \Updownarrow & \\
y^{(9)}_{n+\frac{1}{2}-\beta_7} & y^{(9)}_{n+\frac{1}{2}+\beta_7} & & y^{(9)}_{n+\frac{1}{2}-\beta_8} & y^{(9)}_{n+\frac{1}{2}+\beta_8} & O(h_n^9) \\
 & \Updownarrow & & \Updownarrow & & \\
 & y^{(10)}_{n+\frac{1}{2}-\beta_9} & & y^{(10)}_{n+\frac{1}{2}+\beta_9} & & O(h_n^{10}) \\
y^{(10)}_{n+\frac{1}{2}-\beta_{10}} & y^{(10)}_{n+\frac{1}{2}+\beta_{10}} & y^{(10)}_{n+\frac{1}{2}} & y^{(10)}_{n+\frac{1}{2}-\beta_{11}} & y^{(10)}_{n+\frac{1}{2}+\beta_{11}} & O(h_n^{10})
\end{array} \tag{2.14}$$

We use the same Lobatto 12$^{\text{th}}$ order mesh (2.1) as used for MIRK 10. The coefficients $\beta_{10,11}$ are equivalent to the MIRK 10 $\alpha_{5,6}$. We solve for $\beta_{5,6,7,8,9}$ to give an $O(h_n^{13})$ accurate finite difference scheme, leaving us with the $\beta_{1,2,4}$ to vary. Numerical experiments have been performed on a variety of model problems, and the final error has been minimised by optimising $\beta_{1,2,4}$. More detailed error analysis is needed for MIRK 12 and this will be the subject of a future paper. Constants and Fortran 90 program code for both MIRK 10 & 12 can be found in [2]. For the remainder of this paper we focus our analysis on MIRK 10 and interpolants of order 10, but provide numerical results for MIRK 12 where applicable.

2.1 Accuracy of the MIRK 10 scheme

A detailed error analysis of MIRK 10 gives the local truncation error (A.1). In what follows we will verify that this expression is valid for the following model problems.

$$\left(1+x^2\right)y'' + 4xy' + 2y = 0 \ \text{ where } \ y = \frac{1}{1+x^2} \tag{2.15}$$

$$\text{and } \ y'' - y - y^2 + e^{-2x} = 0 \ \text{ where } \ y = e^{-x}. \tag{2.16}$$

We now outline the error analysis of our MIRK 10 scheme applied to (2.15). First (2.15) is formulated as the system:

$$f(x,Y) = \begin{pmatrix} Y_2 \\ -\frac{1}{1+x^2}\left(4xY_2 + 2Y_1\right) \end{pmatrix} \tag{2.17}$$

Substituting (2.17) into (A.1) gives the following equations:

$$\begin{aligned}
Y_1^{\text{lte}} = \frac{h_n^{11}\left(1+x^2\right)^{-10}}{543642820680960} \Big(\big(&22751427012014\,x^7 - 4857465527933\,x^5 \\
&-4070010302810\,x^3 - \tfrac{9842399897899}{2}\,x^9 - \tfrac{201997864755}{2}\,x\big)\,Y_1 \\
&+\big(\tfrac{341651876047}{4} + \tfrac{148868280858939}{4}\,x^8 - \tfrac{22589985985881}{4}\,x^{10} \\
&\quad - \tfrac{16835420302157}{2}\,x^4 - \tfrac{47149389100565}{2}\,x^6 + \tfrac{3939207369971}{4}\,x^2\big)\,Y_2\Big)
\end{aligned} \tag{2.18a}$$

h_n	O^{n*}	O^{n**}
1	10.779 033 459	11.432 026 354
2^{-1}	11.055 245 670	12.363 978 667
2^{-2}	11.026 385 588	12.860 891 650
2^{-3}	11.007 553 266	12.966 199 422
2^{-4}	11.001 951 457	12.991 607 766
2^{-5}	11.000 491 866	12.997 905 515
2^{-6}	11.000 123 217	12.999 476 601
2^{-7}	11.000 030 820	12.999 869 164
2^{-8}	11.000 007 706	12.999 967 291
2^{-9}	11.000 001 926	12.999 991 823
2^{-10}	11.000 000 481	12.999 997 955
2^{-11}	11.000 000 120	12.999 999 488
2^{-12}	11.000 000 030	12.999 999 872
2^{-13}	11.000 000 007	12.999 999 968

Table 1: Leading order (O^{n*}) and next lowest order (O^{n**}) of the MIRK 10 finite difference scheme applied to $(1+x^2)y''+4xy'+2y=0$.

h_n	O^{n*}	O^{n**}
1	10.872 837 900	13.063 439 773
2^{-1}	10.947 339 460	13.015 984 596
2^{-2}	10.986 135 326	13.004 003 905
2^{-3}	10.996 498 164	13.001 001 460
2^{-4}	10.999 122 429	13.000 250 395
2^{-5}	10.999 780 477	13.000 062 600
2^{-6}	10.999 945 111	13.000 015 650
2^{-7}	10.999 986 277	13.000 003 912
2^{-8}	10.999 996 569	13.000 000 978
2^{-9}	10.999 999 142	13.000 000 244
2^{-10}	10.999 999 785	13.000 000 061
2^{-11}	10.999 999 946	13.000 000 015
2^{-12}	10.999 999 986	13.000 000 003
2^{-13}	10.999 999 996	13.000 000 000

Table 2: Leading order (O^{n*}) and next lowest order (O^{n**}) of the MIRK 10 finite difference scheme applied to $y''-y-y^2+e^{-2x}=0$.

$$\begin{aligned} Y_2^{\text{lte}} = \tfrac{h_n^{11}\left(1+x^2\right)^{-11}}{135910705170240} \Big(\big(&-44396300674 + \tfrac{3149125877181}{4}\,x^2 + 3336435112721\,x^4 \\ &-13841502169549\,x^8 - \tfrac{443277907265}{2}\,x^6 + \tfrac{12661738744269}{4}\,x^{10}\big)\,Y_1 \\ &+\big(\tfrac{6304343370647}{8}\,x^3 - \tfrac{184066518695577}{8}\,x^9 - \tfrac{3013701596181}{8}\,x \\ &+\tfrac{38573186993695}{4}\,x^7 + \tfrac{35560427503695}{4}\,x^5 + \tfrac{29078566305307}{8}\,x^{11}\big)\,Y_2\Big) \end{aligned} \tag{2.18b}$$

For $h_n = 1, 2^{-1}, \ldots, 2^{-14}$, the values $x_n = \frac{1-h_n}{2}$, $x_{n+1} = \frac{1+h_n}{2}$, $y_n = y(x_n)$ and $y_{n+1} = y(x_{n+1})$ are entered into the MIRK 10 finite difference scheme (2.1) giving:

$$\begin{aligned} Y^{n*} = y_n - y_{n+1} + \frac{h_n}{2100}\Big[&\left(372+21\sqrt{15}\right)\left(y'^{(8)}_{n+\frac{1}{2}+\alpha_5} + y'^{(8)}_{n+\frac{1}{2}-\alpha_5}\right) \\ &+\left(372-21\sqrt{15}\right)\left(y'^{(8)}_{n+\frac{1}{2}+\alpha_6} + y'^{(8)}_{n+\frac{1}{2}-\alpha_6}\right) + 512y'^{(8)}_{n+\frac{1}{2}} \\ &+50\left(y'_{n+1}+y'_n\right)\Big]. \end{aligned} \tag{2.19}$$

We also compute

$$Y^{n**} = Y^{n*} - Y^{\text{lte}}\left(x_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}\right). \tag{2.20}$$

From this we calculate the leading order O^{n*} and the next lowest order O^{n**}:

$$\begin{aligned} O^{n*} &= \log_2\left(||Y^{n*}||_2/||Y^{n+1*}||_2\right) \\ O^{n**} &= \log_2\left(||Y^{n**}||_2/||Y^{n+1**}||_2\right) \end{aligned} \tag{2.21}$$

O^{n*} and O^{n**} are tabulated in table 1. The corresponding O^{n*} and O^{n**} terms for problem (2.16) can be found in table 2. As $h_n \to 0$, the orders tend towards 11 and the next lowest orders tend towards 13. This confirms that we have the correct expressions for the leading order error terms for problems (2.15) and (2.16). Additional numerical experiments which verify the order of our MIRK 10 scheme are presented in section 4. In the next section we consider the stability of our scheme.

2.2 Stability of the MIRK 10 scheme

We now examine the stability properties of our MIRK 10 scheme. If we apply (2.1) to the test equation $y' = \lambda y$, we find (where $z = h\lambda$):

$$\frac{y_{n+1}}{y_n} = R(z) = \frac{15\,z^7+466\,z^6+6972\,z^5+63840\,z^4+388080\,z^3+1663200\,z^2+5322240\,z+10644480}{-15\,z^7+466\,z^6-6972\,z^5+63840\,z^4-388080\,z^3+1663200\,z^2-5322240\,z+10644480} \tag{2.22}$$

For A-stability we require $|R(h\lambda)| < 1$, where $\Re(\lambda) < 0$. The MIRK 4[5], 6 & 8 [7] methods have Padé approximants of e^z for their stability functions. From [1], this implies that these methods are A-stable.

Equation (2.22) is not a Padé approximant of e^z, but does exhibit the required properties for our MIRK 10 scheme to be A-stable. For all λ s.t. $\Re(\lambda) = 0$, we see that $|R(h\lambda)| = 1$. Also, as $z \to \infty$, it is apparent that $|R(z)| \to 1$. The denominator of (2.22) has only roots in the open right half plane, implying that $R(z)$ is analytic for all z s.t. $\Re(z) \le 0$. From the maximum principle, we thus deduce that our MIRK 10 scheme is A-stable and that its region of absolute stability is exactly the complex left hand half plane.

3 MIRK 10 Interpolant

One of the major disadvantages of MIRK schemes was originally the fact that their solution was limited to discrete points. This rendered finding event locations such as roots and stationary points complicated. Cash and Wright [8] addressed this issue by introducing an interpolant for MIRK methods. Consideration of the internal points associated with the MIRK finite difference scheme allowed for high order piecewise Hermite-Birkhoff interpolants to be constructed for each mesh point interval. Cash and Moore [6] refined this technique for MIRK 6 & 8 as well as for second order equations of the special commonly occurring forms $D^2Y = f(x, Y)$ and $D^2Y = f(x, Y, DY)$.

We adopt the approach used by Cash and Moore to construct a tenth order interpolant through our mesh points y_n and y_{n+1}. Various bootstrapping formulae are adopted, the constants for which namely $\{s_i, t_i, u_i, \alpha_i\}$ can be found in appendix D. For a tenth order Hermite-Birkhoff interpolant making use of first derivative information, we require at least 2 points of $O(h_n^{\geq 10})$ and 8 points of $O(h_n^{\geq 9})$. From the solution of MIRK 10 we have the following points of interest:

$$\begin{array}{ll} O(h_n^8) & y'^{(8)}_{n+\frac{1}{2}\pm\alpha_5},\ y'^{(8)}_{n+\frac{1}{2}\pm\alpha_6},\ y'^{(8)}_{n+\frac{1}{2}} \\ O(h_n^9) & \left[y'^{(8)}_{n+\frac{1}{2}+\alpha_5} - y'^{(8)}_{n+\frac{1}{2}-\alpha_5}\right],\ \left[y'^{(8)}_{n+\frac{1}{2}+\alpha_6} - y'^{(8)}_{n+\frac{1}{2}-\alpha_6}\right] \\ O(h_n^{10}) & y_{n+1},\ y_n,\ y'_{n+1},\ y'_n \end{array} \tag{3.1}$$

We therefore require at least 4 more approximations of $O(h_n^{\geq 9})$. A Hermite-Birkhoff interpolant $y^{(8)}_{n+\omega}$ can be passed through:

$$\left\{ y_n,\ y_{n+1},\ y'_n,\ y'_{n+1},\ y'^{(8)}_{n+\frac{1}{2}\pm\alpha_5},\ y'^{(8)}_{n+\frac{1}{2}\pm\alpha_6} \right\} \tag{3.2}$$

For $\omega = \pm\frac{\sqrt{297\pm132\sqrt{3}}}{66} = \pm\alpha_{7,8}$, we obtain $O(h_n^{\geq 9})$ points. These can be computed using the following bootstrapping scheme:

Order 9 points:

$$\begin{aligned} y^{(9)}_{n+\frac{1}{2}\pm\alpha_7} = s_1^{\pm} y_n + s_1^{\mp} y_{n+1} \mp h_n \Big(& s_2^{\pm} y'_n + s_2^{\mp} y'_{n+1} + s_3^{\pm} y'^{(8)}_{n+\frac{1}{2}-\alpha_5} \\ & + s_3^{\mp} y'^{(8)}_{n+\frac{1}{2}+\alpha_5} + s_4^{\pm} y'^{(8)}_{n+\frac{1}{2}-\alpha_6} + s_4^{\mp} y'^{(8)}_{n+\frac{1}{2}+\alpha_6} \Big), \end{aligned} \tag{3.3}$$

$$\begin{aligned} y^{(9)}_{n+\frac{1}{2}\pm\alpha_8} = t_1^{\pm} y_n + t_1^{\mp} y_{n+1} \mp h_n \Big(& t_2^{\pm} y'_n + t_2^{\mp} y'_{n+1} + t_3^{\pm} y'^{(8)}_{n+\frac{1}{2}-\alpha_5} \\ & + t_3^{\mp} y'^{(8)}_{n+\frac{1}{2}+\alpha_5} + t_4^{\pm} y'^{(8)}_{n+\frac{1}{2}-\alpha_6} + t_4^{\mp} y'^{(8)}_{n+\frac{1}{2}+\alpha_6} \Big). \end{aligned} \tag{3.4}$$

Unfortunately we are unable to obtain an order 10 Hermite-Birkhoff interpolant $y^{(10)}_{n+\omega}$ from the following points:

$$\left\{ y_n,\ \ y_{n+1},\ \ y'_n,\ \ y'_{n+1},\ \ \left[y'^{(8)}_{n+\frac{1}{2}+\alpha_{5,6}} - y'^{(8)}_{n+\frac{1}{2}-\alpha_{5,6}} \right],\ \ y'^{(9)}_{n+\frac{1}{2}\pm\alpha_7},\ \ y'^{(9)}_{n+\frac{1}{2}\pm\alpha_8} \right\} \tag{3.5}$$

It is possible, however, to use these data points to obtain an order 10 interpolant at $\omega = \pm\alpha_9 = \pm\frac{\sqrt{33}}{22}$. We therefore perform the following additional bootstrapping:
Order 10 points:

$$\begin{aligned} y^{(10)}_{n+\frac{1}{2}\pm\alpha_9} = & u_1^{\pm} y_n + u_1^{\mp} y_{n+1} \mp h_n \Big(u_2^{\pm} y'_n + u_2^{\mp} y'_{n+1} \\ & \pm u_3 \left[y'^{(8)}_{n+\frac{1}{2}+\alpha_5} - y'^{(8)}_{n+\frac{1}{2}-\alpha_5} \right] \pm u_4 \left[y'^{(8)}_{n+\frac{1}{2}+\alpha_6} - y'^{(8)}_{n+\frac{1}{2}-\alpha_6} \right] \\ & + u_5^{\pm} y'^{(9)}_{n+\frac{1}{2}-\alpha_7} + u_5^{\mp} y'^{(9)}_{n+\frac{1}{2}+\alpha_7} + u_6^{\pm} y'^{(9)}_{n+\frac{1}{2}-\alpha_8} + u_6^{\mp} y'^{(9)}_{n+\frac{1}{2}+\alpha_8} \Big). \end{aligned} \tag{3.6}$$

We are now able to construct an order 10 interpolant through the following points:

$$\left\{ y_n,\ \ y_{n+1},\ \ y'_n,\ \ y'_{n+1},\ \ y'^{(9)}_{n+\frac{1}{2}\pm\alpha_7},\ \ y'^{(9)}_{n+\frac{1}{2}\pm\alpha_8},\ \ y'^{(10)}_{n+\frac{1}{2}\pm\alpha_9} \right\} \tag{3.7}$$

The interpolant $y^{(10)}_{n+\omega}$, $\omega \in [0,1]$ takes the form:

$$\begin{aligned} y^{(10)}_{n+\omega} = A(\omega) y_{n+1} + (1 - A(\omega))\, y_n + h_n \Big\{ & B(\omega) y'_{n+1} - B(1-\omega) y'_n \\ & + C(\omega) y'^{(9)}_{n+\frac{1}{2}+\alpha_7} - C(1-\omega) y'^{(9)}_{n+\frac{1}{2}-\alpha_7} \\ & + D(\omega) y'^{(9)}_{n+\frac{1}{2}+\alpha_8} - D(1-\omega) y'^{(9)}_{n+\frac{1}{2}-\alpha_8} \\ & + E(\omega) y'^{(10)}_{n+\frac{1}{2}+\alpha_9} - E(1-\omega) y'^{(10)}_{n+\frac{1}{2}-\alpha_9} \Big\} \end{aligned} \tag{3.8}$$

where:

$$\begin{aligned} A(\omega) = & -\omega^2 \left(-210 + 2590\,\omega - 14070\,\omega^2 + 41664\,\omega^3 - 71610\,\omega^4 \right. \\ & \left. + 71280\,\omega^5 - 38115\,\omega^6 + 8470\,\omega^7 \right) \end{aligned} \tag{3.9}$$

$$\begin{aligned} B(\omega) = & \frac{1}{24}\,\omega^2\,(\omega - 1) \left(8833\,\omega^6 - 30371\,\omega^5 + 42097\,\omega^4 - 30008\,\omega^3 \right. \\ & \left. + 11620\,\omega^2 - 2354\,\omega + 207 \right) \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
C(\omega) = & \frac{1}{95568} \Big(1445466\,\omega^5 + \Big(91476\,\alpha_8\sqrt{11} - 176418\,\alpha_8\sqrt{33} - 3613665\Big)\,\omega^4 \\
& + \Big(-182952\,\alpha_8\sqrt{11} + 352836\,\alpha_8\sqrt{33} + 8712\,\sqrt{3} + 3441240\Big)\,\omega^3 \\
& + \Big(-13068\,\sqrt{3} - 265650\,\alpha_8\sqrt{33} - 1548195 + 157212\,\alpha_8\sqrt{11}\Big)\,\omega^2 \\
& + \Big(89232\,\alpha_8\sqrt{33} + 334158 - 65736\,\alpha_8\sqrt{11} + 8316\,\sqrt{3}\Big)\,\omega - 29502 \\
& -1980\,\sqrt{3} + 11880\,\alpha_8\sqrt{11} - 12672\,\alpha_8\sqrt{33}\Big)\,(\omega-1)^2\,\omega^2\,\Big(148+\sqrt{3}\Big)
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
D(\omega) = & \frac{1}{95568}\,\omega^2\,(\omega-1)^2\,\Big(\sqrt{3}-148\Big)\,\Big(-1445466\,\omega^5 \\
& + \Big(3613665 + 1221858\,\alpha_8 - 431244\,\alpha_8\sqrt{3}\Big)\,\omega^4 \\
& + \Big(-2443716\,\alpha_8 + 862488\,\alpha_8\sqrt{3} + 8712\,\sqrt{3} - 3441240\Big)\,\omega^3 \\
& + \Big(-13068\,\sqrt{3} - 590964\,\alpha_8\sqrt{3} + 1548195 + 1762002\,\alpha_8\Big)\,\omega^2 \\
& + \Big(159720\,\alpha_8\sqrt{3} - 334158 + 8316\,\sqrt{3} - 540144\,\alpha_8\Big)\,\omega \\
& +29502 - 15048\,\alpha_8\sqrt{3} - 1980\,\sqrt{3} + 66528\,\alpha_8\Big)
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
E(\omega) = & \frac{121}{48}\,\omega^2\,(\omega-1)^2\,\Big(-242\,\omega^5 + \Big(33\,\sqrt{33}+605\Big)\,\omega^4 - \Big(66\,\sqrt{33}+528\Big)\,\omega^3 \\
& + \Big(49\,\sqrt{33}+187\Big)\,\omega^2 - \Big(16\,\sqrt{33}+22\Big)\,\omega + 2\,\sqrt{33}\Big)
\end{aligned}
\tag{3.13}
$$

Plots of $A(\omega)$,...,$E(\omega)$ can be found in figure 2. In order to form (3.8), 6 function evaluations are required. Next, we explore the accuracy of the interpolant.

3.1 Accuracy of the Interpolant

To verify the accuracy of (3.8) for a particular non-linear problem we consider Enright's model problem [10]:

$$
y' = -\frac{y^3}{2} \qquad y(0) = 1,\;\; y(x) = \frac{1}{\sqrt{1+x}} \tag{3.14}
$$

For $h_n = 1, \frac{1}{2}, \ldots, \frac{1}{1024}$; we solve (3.14) for the two mesh points y_n and y_{n+1} where $x_n = 1 - \frac{h_n}{2}$ and $x_{n+1} = 1 + \frac{h_n}{2}$. The interpolant (3.8) is then constructed through (3.7). The order of interpolation is checked by computing:

$$
\epsilon^{(10)}(\omega) = \frac{y\left(1 + h_n\left(\omega - \frac{1}{2}\right)\right) - y^{(10)}_{n+\omega}}{h_n^{10}} = \frac{y_{\text{exact}}(\omega) - y^{(10)}_{n+\omega}}{h_n^{10}}. \tag{3.15}
$$

A plot of $\epsilon^{(10)}(\omega)$ for $\omega \in [0,1]$ and $h_n = 1, \frac{1}{2}, \ldots, \frac{1}{1024}$ can be found in figure 3. This shows $\epsilon^{(10)}(\omega)$ tending towards a a bold curve as $h_n \to 0$. We thus deduce that the leading error term of $y^{(10)}_{n+\omega}$ is $O(h_n^{10})$. We can further verify this by cancelling out the leading $O(h_n^{10})$ error term which we now calculate.

The $O(h_n^{10})$ error term of (3.8) is composed of the interpolant truncation error $\epsilon_{\text{lte}}(\omega)$ and the truncation errors of our order 9 points:

$$
y'\left(x_n + h_n\left(\frac{1}{2} \pm \alpha_{7,8}\right)\right) - y'^{(9)}_{n+\frac{1}{2}\pm\alpha_{7,8}} = \epsilon_{\pm\alpha_{7,8}}(\omega) + O(h_n^{10}) \tag{3.16}
$$

The general form for $\epsilon_{\text{lte}}(\omega)$ is:

$$\begin{aligned}\epsilon_{\text{lte}}(\omega) = &-\frac{h_n^{10}}{7903526400}\,\omega^2\left(2178\,\omega^6 - 6534\,\omega^5 + 8019\,\omega^4 - 5148\,\omega^3\right.\\ &\left.+1835\,\omega^2 - 350\,\omega + 30\right)(\omega-1)^2\,\frac{\mathrm{d}^{10}y_n}{\mathrm{d}x^{10}}\left(1 + h_n\left(\omega - \frac{1}{2}\right)\right) + O(h_n^{11})\end{aligned} \tag{3.17}$$

Whilst the general form for $\epsilon_{\pm\alpha_{7,8}}(x)$ can be found in (B.1) and (B.2); for our model problem (3.14) we have:

$$\epsilon_{\pm\alpha_7}(\omega) = \mp\frac{\alpha_7\left(7012169047902608\,\sqrt{3} + 11545612665589385\right)h_n^9}{150543181326260895744\left(2 + h_n\left(\omega - \frac{1}{2}\right)\right)^{21/2}} \tag{3.18}$$

$$\epsilon_{\pm\alpha_8}(\omega) = \pm\frac{\alpha_8\left(7012169047902608\,\sqrt{3} - 11545612665589385\right)h_n^9}{150543181326260895744\left(2 + h_n\left(\omega - \frac{1}{2}\right)\right)^{21/2}} \tag{3.19}$$

The complete $O(h_n^{10})$ error term is:

$$\begin{aligned}\epsilon_{10}(\omega) =&h_n\left\{C(\omega)\epsilon_{+\alpha_7}(\omega) - C(1-\omega)\epsilon_{-\alpha_7}(\omega)\right.\\ &\left.+D(\omega)\epsilon_{+\alpha_8}(\omega) - D(1-\omega)\epsilon_{-\alpha_8}(\omega)\right\} - \epsilon_{\text{lte}}(\omega)\end{aligned} \tag{3.20}$$

We plot the following in figure 4:

$$\epsilon^{(11)}(\omega) = \frac{y_{\text{exact}}(\omega) - y_{n+\omega}^{(10)} + \epsilon_{10}(\omega)}{h_n^{11}} \tag{3.21}$$

As $h_n \to 0$, $\epsilon^{(11)}(\omega)$ tends towards the bold curve shown in figure 4 confirming that (3.20) contains all the $O(h_n^{10})$ error terms of our interpolant. To gauge the effectiveness of our interpolant we perform numerical experiments on the Orr-Sommerfeld problem in the next section.

4 Case Study: Orr-Sommerfeld

For a test case we consider the Orr-Sommerfeld equation

$$\begin{aligned} y^{iv} = 2\alpha^2 y'' - \alpha^4 y + \mathrm{i}\alpha R\left(\left(y'' - \alpha^2 y\right)\left(1 - x^2 - \lambda\right) + 2y\right), \quad & x \in [-1, 1]\\ & y, \lambda \in \mathbb{C}\\ & \alpha, R \in \mathbb{R}\end{aligned} \tag{4.1}$$

together with the boundary conditions

$$y(-1) = y'(-1) = y(1) = y'(1) = 0. \tag{4.2}$$

We formulate (4.1) as real system of dimension 8. Here $\alpha = 1$ and $R = 10000$ are fixed, whilst the real and imaginary parts of λ are solved for as parameters. Our target solution is mode 1 symmetric with eigenvalue $\lambda \approx 0.2375 + 0.0037\mathrm{i}$. Cash, Garcia and Moore present a solution to this problem using MIRK 4, 6 & 8 methods in [4]. For 25 mesh points we solve (4.1) using the MIRK 10 scheme then construct a set of 24 piecewise interpolants through the mesh points. Plots of the relative error of the solution can be found in figures 5 & 6, the relative error $\epsilon(x)$ being measured in the following way:

$$\epsilon(x) = ||Y(x) - Y_i^*||_2/||Y_i^*||_2 \tag{4.3}$$

where $Y(x)$ is obtained from interpolation and Y_i^* is a mesh point from a high accuracy solution of 3073 points. Figure 6 also illustrates the characteristic MIRK 10 interpolant error curve found in figure 3. As x increases, our mesh point accuracy decreases due to a rapid change to the solution behaviour at $x = 1$. The interpolant error is also adversely affected as x increases; this can be deduced from the fact that the error curve does not follow the pattern found in figure 3. The uniform distribution of mesh points is likely to be responsible for both of these problems.

A high accuracy solution to the Orr-Sommerfeld problem using Chebyshev polynomial approximation by Orszag can be found in [12]. Orszag's approach is superior to finite difference methods as far as accuracy is concerned due to the fact that it is not restricted to a finite order of accuracy. Nevertheless, we perform a comparison between Orsag's method and our MIRK 10 scheme. With 13 even-degree Chebyshev polynomials, Orszag's method is roughly 40 times less accurate at computing λ than a 13 mesh point MIRK 10 solution. Increasing the number of points/polynomials leads to a greater increase in accuracy with Orszag's method, however. At 25 points, Orszag's method produces a λ which is roughly 10 times more accurate than is provided by MIRK 10. A comparison of Orszag's method with MIRK 10 and MIRK 12 can be found in table 3.

Our method does have one major advantage over Orszag's, namely its generality. MIRK 10 can be applied to any $DY = f(x, Y)$ system, the majority of the effort required for obtaining a solution lies in setting up a good initial guess. For this, the authors have observed that solving a MIRK 6 [7] scheme is often the best way to deduce an initial guess for MIRK 10. We now modify our formulation of (4.1) to allow for the computation of the critical Reynolds number R_c. If we consider R as a function of α; R_c is defined to be the value of R such that $\frac{\partial R}{\partial \alpha} = 0$ with λ real. Taking partial derivatives of the 8 dimensional system with respect to α gives a system of dimension 16 with the four parameters R, α, λ and $\partial\lambda/\partial\alpha$. Solving this for 1025 mesh points we obtain the values:

$$\begin{aligned} R_c &= 5772.22243772212601111 \\ \alpha &= 1.02062227094822711098 \end{aligned} \tag{4.4}$$

Figure 1 illustrates the relative error at the end points:

$$\text{error} = \frac{||Y_l - Y_l^*||_2}{||Y_l^*||_2} + \frac{||Y_r - Y_r^*||_2}{||Y_r^*||_2} \qquad \text{where} \qquad \begin{aligned} Y_{l,r} &= \text{left/right mesh point.} \\ Y_{l,r}^* &= \text{left/right accurate mesh point.} \end{aligned} \tag{4.5}$$

From this it is apparent that our MIRK 10 method exhibits order 10 accuracy, agreeing with our local truncation error analysis in section 2.1. Also our MIRK 12 method can be seen to have order 12 accuracy. We perform additional numerical testing on our MIRK 10 & 12 methods in the next section.

5 Numerical Tests

In [14, 9] Wright proposed a set of 32 singular perturbation problems suitable for testing two point boundary value problem solvers. These problems are of the form:

$$\epsilon \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,\ y) \qquad a \leq x \leq b, \qquad g(y(a),\ y(b)) = 0. \tag{5.1}$$

Reducing ϵ increases the stiffness of the problems and so the test problems provide a challenge for our MIRK 10 & 12 schemes. The error when solving the problems can be conveniently measured in two ways (where M is the number of mesh points):

$$E_{\text{avg}} = \frac{1}{M}\sum_{i=1}^{M} ||Y_i - Y_i^*||_2 \qquad E_{\max} = \max_{i=1,\ldots,M} ||Y_i - Y_i^*||_2 \qquad \begin{aligned} Y_i &= \text{numerical mesh point solution} \\ Y_i^* &= \text{accurate mesh point solution} \end{aligned} \tag{5.2}$$

E_{avg} gives a measure of the global error of the solver, and is used by the our routines to internally verify the order of accuracy when solving the test problems. A measure of the worst case error is $E_{\max}$; from this and E_{avg} we can deduce wether or not there's a large discrepancy between the accuracy of the solution at different mesh points. Accuracy tests were carried out for MIRK 6 & 8 (Cash-Singhal) [7] , MIRK 8 (Cash 2000) [3], MIRK 10 and MIRK 12 with uniform meshes of 1025 points. Where analytic solutions to the test problems exist, they were made use of. For the test problems with no analytic solution, an "accurate" mesh of 2049 points was computed, and used to measure the error. The results from these tests can be found in tables 4 and 5. The results are largely as expected. We were able to solve all 32 problems with all methods and we note the tremendous accuracy achieved by the high order methods. The ratio $E_{\max}/E_{\text{avg}}$ is below 10^2 for the test problems implying that the accuracy along the mesh does not vary severely. As can be seen from the tables, MIRK 10 achieves 12^{th} order accuracy for some problems. MIRK 10 is $\approx 10^5$ times more accurate than the MIRK 8 schemes (a saving of roughly 2 mesh doubles) and MIRK 12 is $\approx 10^{10}$ times more accurate than the MIRK 8 schemes (a saving of roughly 4 mesh doubles). This represents a significant saving in both memory and time when computing solutions to the test problems.

6 Conclusions & Further Research

Our derivation of MIRK 10 & 12 can be summarised as bootstrapping with Hermite-Birkhoff interpolants then choosing the relevant points α_i to cancel out the leading order error terms. For MIRK 10 we can calculate 5 out of the 6 points directly from the order conditions. There is, however, no obvious choice for our α_1. We can cancel out certain terms of the local truncation error, but are unable to go up an order of accuracy or cancel out any of the coefficients of the scheme. The final value we chose for α_1 was to allow for the simplification of the analytic expression for the coefficients. Similarly with MIRK 12, we have 3 points $\beta_{1,2,4}$ for which there are no obvious choices. Numerical experiments have been carried out on a set of model problems to find optimal points. An increase in accuracy of roughly 3 decimal places has been realised so far. More detailed analysis will need to be applied to the error terms, to identify the reasons for the increase. For MIRK 10 & 12 it may even be practical for the solver to perform some optimisation itself to choose the free parameters, thereby tailoring the method to the target problem.

From the numerical experiments in this paper, we see that MIRK 10 & 12 give high accuracy solutions to all the problems. This accuracy comes at a price; the need for a better initial guess for use in the Newton iteration schemes. We have found through practical experience that MIRK 6 provides a "robust" (likely to converge to a solution when given a poor initial guess) way of determining a good initial guess which can be used for MIRK 10 & 12. Indeed we have found MIRK 6 to be more robust than MIRK 4 in some circumstances. The reasons for this are not clear, and is a subject worth further investigation.

We have provided piecewise Hermite-Birkhoff interpolants of accuracy $O(h_n^{10})$ which are explicit. For non-stiff problems, these provide a high accuracy way of determining event locations once a mesh has been computed. Where the problem stiffness increases, it may be advisable to adopt implicit interpolants [3]. These have the advantage of possessing a high accuracy, but are computationally more expensive to evaluate than explicit schemes. The MIRK 10 interpolant presented in this paper requires 6 function evaluations. The computational cost of computing each of these is normally minimal, but for meshes with a large number of points this cost will become significant. It is not clear how to reduce the number of function evaluations required to form an order 10 interpolant. This is an area worthy of further investigation. We have yet to derive an $O(h_n^{12})$ interpolant, the final 2 stages of MIRK 12 provide points with a sufficient accuracy for the construction of an $O(h_n^{10})$ accurate interpolant. Additional bootstrapping will be required to get to $O(h_n^{12})$ accuracy.

It is worth noting that throughout this paper we have worked exclusively with uniformly spaced meshes. The principal reason for this was to provide a fair comparison between the existing MIRK methods. As MIRK methods rely on internal points at different $x_{n+\frac{1}{2}\pm\alpha_i}$ values, adaptive mesh algorithms based on one MIRK method will likely give different meshes when applied to other MIRK methods. Silva [13], presents a mesh selection algorithm based on deferred correction. If one wishes to compute an efficient MIRK 10 solution to a BVP they are encouraged to use the MIRK 10 interpolant as a basis for an adaptive mesh algorithm as part of a deferred correction scheme and we hope to report on this in the future.

A MIRK 10 Local Truncation Error

$$
\begin{aligned}
y^{\mathrm{lte}} = \frac{h_n^{11}}{15782168541496541118400} \Bigg\{ & 4279905142695 \Bigg[\frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \\
& +2 \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^3 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} - \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} - \left(\frac{\partial f}{\partial y}\right)^5 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \right) \Bigg] \frac{\mathrm{d}^4 y}{\mathrm{d}x^4} \\
& +2680209 \Bigg[-568832 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} + 615051 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^3 \\
& +736170 \left(-\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^4 + \left(\frac{\partial f}{\partial y}\right)^6 \right) - 795728 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \\
& +1137664 \left(-\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^3 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y}\right)^2 \frac{\partial f}{\partial y} \right) \\
& -1230102 \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^3 + 1591456 \Bigg(\left(\frac{\partial f}{\partial y}\right)^4 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \\
& -\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \Bigg) + 1966272 \left(\left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y}\right)^2 \left(\frac{\partial f}{\partial y}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^2 \right) \Bigg] \frac{\mathrm{d}^5 y}{\mathrm{d}x^5} \\
& +7 \Bigg[128557738496 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} + 444383370816 \left(-\left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y}\right)^2 \frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \right) \\
& +182002266918 \left(-\left(\frac{\partial f}{\partial y}\right)^5 + \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^3 \right) + 262381103898 \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^2 \\
& +257115476992 \left(-\frac{\partial f}{\partial y} \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^3 \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} + \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \right) \\
& -131190551949 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} \left(\frac{\partial f}{\partial y}\right)^2 \Bigg] \frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + 2263612 \Bigg[-50720 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{\partial f}{\partial y} + 101440 \frac{\partial f}{\partial y} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \\
& +257147 \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} - \left(\frac{\partial f}{\partial y}\right)^3 \right) \Bigg] \frac{\mathrm{d}^8 y}{\mathrm{d}x^8} + 90189092916 \left[-\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^2 \right] \frac{\mathrm{d}^9 y}{\mathrm{d}x^9} \Bigg\}
\end{aligned} \tag{A.1}
$$

B LTE of Principal Interpolant Error Terms

$$\epsilon_{\pm\alpha_7}(x) = \mp \frac{h_n^9 \sqrt{297-132\sqrt{3}}}{4951429920343946988748800} \left(\frac{\partial f}{\partial y}\right)^2 \Bigg\{ \left(68478482283120\sqrt{3} - 295313454845955\right) \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \left(-131725153714896\sqrt{3} - 308734837271847\right) \left(\frac{\partial f}{\partial y}\right)^3 \frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \left(-24393418334208\sqrt{3} + 105196616566272\right) \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \left(-34123477554432\sqrt{3} + 147157496953488\right) \frac{\partial f}{\partial y}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \left(14398466711552\sqrt{3} - 62093387693568\right) \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + \left(78627166053072\sqrt{3} + 178460329632711\right) \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + \left(-1735789588480\sqrt{3} - 18175805930448\right) \frac{\partial f}{\partial y}\frac{\mathrm{d}^7 y}{\mathrm{d}x^7} + \left(14513193610240\sqrt{3} + 28087115002752\right) \frac{\mathrm{d}^8 y}{\mathrm{d}x^8} \Bigg\} \quad \text{(B.1)}$$

$$\epsilon_{\pm\alpha_8}(x) = \mp \frac{h_n^9 \sqrt{297+132\sqrt{3}}}{4951429920343946988748800} \left(\frac{\partial f}{\partial y}\right)^2 \Bigg\{ \left(68478482283120\sqrt{3} + 295313454845955\right) \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \left(-131725153714896\sqrt{3} + 308734837271847\right) \left(\frac{\partial f}{\partial y}\right)^3 \frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \left(-24393418334208\sqrt{3} - 105196616566272\right) \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \left(-34123477554432\sqrt{3} - 147157496953488\right) \frac{\partial f}{\partial y}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} + \left(14398466711552\sqrt{3} + 62093387693568\right) \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y}\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + \left(-78627166053072\sqrt{3} + 178460329632711\right) \left(\frac{\partial f}{\partial y}\right)^2 \frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + \left(-1735789588480\sqrt{3} + 18175805930448\right) \frac{\partial f}{\partial y}\frac{\mathrm{d}^7 y}{\mathrm{d}x^7} + \left(14513193610240\sqrt{3} - 28087115002752\right) \frac{\mathrm{d}^8 y}{\mathrm{d}x^8} \Bigg\} \quad \text{(B.2)}$$

C Coefficients for the MIRK 10 scheme

$$\alpha_1 = \frac{1}{4} \qquad \alpha_2 = \frac{1}{212914}\sqrt{5294425981} \qquad \alpha_3 = \frac{1}{114}\sqrt{2185}$$
$$\alpha_4 = \frac{1}{114}\sqrt{1653} \qquad \alpha_5 = \frac{1}{66}\sqrt{495-66\sqrt{15}} \qquad \alpha_6 = \frac{1}{66}\sqrt{495+66\sqrt{15}}$$

Number of Curves/Points	$\lvert\lambda-\lambda^*\rvert$		
	Orszag	MIRK 10	MIRK 12
13	1.936e-3	4.680e-5	3.442e-7
14	6.234e-4	2.258e-5	1.473e-7
16	9.491e-5	6.205e-6	3.103e-8
19	5.407e-6	1.071e-6	4.076e-9
22	2.587e-7	2.340e-7	6.923e-10
25	8.842e-9	6.143e-8	1.460e-10
28 †	1.334e-9	1.861e-8	3.641e-11

Table 3: Absolute error when computing λ using Orszag's method, MIRK 10 & MIRK 12. Orszag's results were obtained from [12]

†this row will have a less accurate result for Orszag's error than the others due to the fact that we are using all 8 digits Orszag provided for λ.

$$a_1^- = \tfrac{27}{32} \qquad a_1^+ = \tfrac{5}{32} \qquad a_2^- = \tfrac{9}{64} \qquad a_2^+ = -\tfrac{3}{64}$$

$$\begin{aligned}
b_1^\pm &= \tfrac{1}{2} \mp \tfrac{134819}{22666185698}\sqrt{5294425981} \qquad b_3^\pm = \mp\tfrac{536268696}{11333092849} \\
b_2^\pm &= \tfrac{14181}{22666185698}\sqrt{5294425981} \mp \tfrac{973398021}{22666185698} \\
c_1^\pm &= \tfrac{1}{2} \mp \tfrac{154085}{13678632}\sqrt{2185} \qquad c_2^\pm = \tfrac{71731079}{122511587616}\sqrt{2185} \mp \tfrac{618856}{21824559} \\
c_3^\pm &= \tfrac{1538286841}{2327720164704}\sqrt{2185} \mp \tfrac{274547}{1085400792747}\sqrt{5294425981} \\
d_1^\pm &= \tfrac{1}{2} \mp \tfrac{8053}{656298}\sqrt{1653} \qquad d_2^\pm = \tfrac{1001}{2625192}\sqrt{1653} \mp \tfrac{7}{456} \\
d_3^\pm &= \tfrac{21}{15352}\sqrt{1653} \mp \tfrac{21}{17480}\sqrt{2185}
\end{aligned}$$

$$e_1 = \tfrac{827}{12544} \qquad e_2 = -\tfrac{1539}{288512}\sqrt{2185} \qquad e_3 = \tfrac{57}{6272}\sqrt{1653}$$

$$\begin{aligned}
f_1^\pm &= \tfrac{1}{2} \mp \tfrac{11085\,\alpha_5}{5324} \mp \tfrac{1505}{31944}\sqrt{15}\alpha_5 \\
f_2^\pm &= \mp\tfrac{3191}{149072} \mp \tfrac{2225}{1565256}\sqrt{15} + \tfrac{40085\,\alpha_5}{447216} + \tfrac{1979}{894432}\sqrt{15}\alpha_5 \\
f_3 &= \tfrac{171}{24000592}\sqrt{1311}\left(7\sqrt{15} - 124\right) \\
f_4^\pm &= \mp\tfrac{437}{308792}\sqrt{1653} \mp \tfrac{6745}{15130808}\sqrt{24795} + \tfrac{1543275\,\alpha_5}{4323088} + \tfrac{267501}{8646176}\sqrt{15}\alpha_5 \\
f_5 &= -\tfrac{2\,\alpha_5\left(-994+101\sqrt{15}\right)}{10527}
\end{aligned}$$

$$\begin{aligned}
r_1 &= -\tfrac{121}{7168} \qquad r_2 = -\tfrac{1083}{207872}\sqrt{1653} \\
r_3 &= \tfrac{3\,\alpha_5\left(7\sqrt{15}+124\right)}{512} \qquad r_4 = -\tfrac{3\,\alpha_6\left(7\sqrt{15}-124\right)}{512}
\end{aligned}$$

$$\begin{aligned}
g_1^\pm &= \tfrac{1}{2} \mp \tfrac{11085\,\alpha_6}{5324} \pm \tfrac{1505}{31944}\sqrt{15}\alpha_6 \\
g_2^\pm &= \mp\tfrac{3191}{149072} \pm \tfrac{2225}{1565256}\sqrt{15} + \tfrac{40085\,\alpha_6}{447216} - \tfrac{1979}{894432}\sqrt{15}\alpha_6 \\
g_3 &= \tfrac{171}{24000592}\sqrt{1311}\left(7\sqrt{15} + 124\right) \\
g_4^\pm &= \mp\tfrac{437}{308792}\sqrt{1653} \pm \tfrac{6745}{15130808}\sqrt{24795} + \tfrac{1543275\,\alpha_6}{4323088} - \tfrac{267501}{8646176}\sqrt{15}\alpha_6 \\
g_5 &= \tfrac{2\,\alpha_6\left(994+101\sqrt{15}\right)}{10527}
\end{aligned}$$

		Error - E_{avg}				
Problem	ϵ	MIRK 12	MIRK 10	Cash-Singhal 8	Cash 2000 8	MIRK 6
1 *	0.0010	0.12E-30	0.31E-29	0.78E-25	0.34E-18	0.86E-14
2 *	0.0100	0.12E-24	0.31E-23	0.78E-20	0.34E-14	0.86E-11
3	0.0500	0.81E-30	0.33E-24	0.70E-20	0.49E-20	0.40E-16
4 *	0.0250	0.56E-26	0.14E-24	0.54E-21	0.35E-15	0.13E-11
5	0.0100	0.16E-28	0.24E-23	0.35E-19	0.17E-19	0.76E-16
6	0.0220	0.12E-30	0.11E-24	0.11E-19	0.87E-20	0.40E-15
7	0.0250	0.38E-31	0.38E-25	0.17E-20	0.10E-20	0.37E-16
8 *	0.0100	0.12E-24	0.31E-23	0.78E-20	0.34E-14	0.86E-11
9	0.0550	0.53E-30	0.99E-24	0.30E-18	0.41E-18	0.22E-13
10	0.0220	0.75E-31	0.78E-25	0.98E-20	0.84E-20	0.40E-15
11	0.0010	0.16E-31	0.20E-24	0.27E-20	0.10E-19	0.14E-15
12 *	0.0025	0.10E-29	0.13E-25	0.43E-21	0.11E-17	0.18E-13
13 *	0.0025	0.10E-29	0.13E-25	0.43E-21	0.11E-17	0.18E-13
14 *	0.0025	0.20E-29	0.13E-25	0.43E-21	0.22E-17	0.35E-13
15	0.0050	0.12E-29	0.14E-23	0.32E-18	0.32E-17	0.12E-12
16 *	0.0525	0.18E-28	0.46E-27	0.13E-22	0.64E-16	0.18E-11
17	0.0005	0.47E-30	0.75E-24	0.20E-18	0.13E-18	0.35E-14
18 *	0.0100	0.12E-24	0.31E-23	0.78E-20	0.34E-14	0.86E-11
19	0.0300	0.86E-27	0.67E-22	0.33E-17	0.92E-17	0.65E-13
20	0.0500	0.32E-30	0.18E-24	0.10E-19	0.17E-19	0.38E-15
21	0.0008	0.97E-28	0.49E-22	0.15E-17	0.93E-17	0.15E-12
22	0.0250	0.10E-28	0.91E-23	0.34E-18	0.29E-17	0.42E-13
23	5.0000	0.18E-25	0.65E-21	0.15E-17	0.20E-16	0.10E-13
24	0.0300	0.85E-30	0.74E-24	0.90E-19	0.13E-18	0.19E-14
25	0.0025	0.35E-30	0.17E-24	0.87E-20	0.76E-20	0.13E-15
26	0.0200	0.43E-26	0.52E-20	0.12E-15	0.30E-15	0.12E-11
27	0.0200	0.43E-26	0.52E-20	0.12E-15	0.30E-15	0.12E-11
28	0.0300	0.45E-29	0.27E-22	0.15E-17	0.39E-17	0.41E-13
29	0.0150	0.25E-29	0.17E-23	0.98E-19	0.16E-18	0.19E-14
30	0.0420	0.24E-31	0.31E-25	0.51E-20	0.84E-20	0.28E-15
31	0.0250	0.21E-30	0.45E-25	0.43E-20	0.20E-18	0.29E-14
32	100.0000	0.41E-30	0.58E-24	0.12E-18	0.96E-19	0.89E-14

Table 4: The 32 test problems solved using MIRK methods of orders 6,8,10 & 12. E_{avg} (5.2) was used to measure the error. The problems were all solved with uniform meshes of 1025 points.

*This problem is of the form $DY = AY + b$, therefore MIRK 10 will have order 12 accuracy and Cash-Singhal 8 will have order 10.

		Error - $E_{\max}$				
Problem	ϵ	MIRK 12	MIRK 10	Cash-Singhal 8	Cash 2000 8	MIRK 6
1 *	0.0010	0.14E-29	0.36E-28	0.91E-24	0.40E-17	0.10E-12
2 *	0.0100	0.45E-23	0.11E-21	0.29E-18	0.13E-12	0.32E-09
3	0.0500	0.25E-29	0.82E-24	0.16E-19	0.11E-19	0.75E-16
4 *	0.0250	0.17E-24	0.43E-23	0.16E-19	0.11E-13	0.39E-10
5	0.0100	0.53E-28	0.91E-23	0.77E-19	0.40E-19	0.13E-15
6	0.0220	0.31E-30	0.31E-24	0.34E-19	0.33E-19	0.15E-14
7	0.0250	0.12E-30	0.11E-24	0.41E-20	0.30E-20	0.91E-16
8 *	0.0100	0.45E-23	0.11E-21	0.29E-18	0.13E-12	0.32E-09
9	0.0550	0.70E-29	0.69E-23	0.12E-17	0.16E-17	0.86E-13
10	0.0220	0.29E-30	0.30E-24	0.34E-19	0.32E-19	0.15E-14
11	0.0010	0.34E-31	0.31E-24	0.42E-20	0.16E-19	0.22E-15
12 *	0.0025	0.15E-28	0.19E-25	0.67E-21	0.16E-16	0.26E-12
13 *	0.0025	0.15E-28	0.19E-25	0.67E-21	0.16E-16	0.26E-12
14 *	0.0025	0.15E-28	0.19E-25	0.67E-21	0.16E-16	0.26E-12
15	0.0050	0.36E-29	0.45E-23	0.11E-17	0.96E-17	0.36E-12
16 *	0.0525	0.56E-28	0.14E-26	0.40E-22	0.20E-15	0.55E-11
17	0.0005	0.39E-29	0.42E-23	0.61E-18	0.71E-18	0.33E-13
18 *	0.0100	0.45E-23	0.11E-21	0.29E-18	0.13E-12	0.32E-09
19	0.0300	0.26E-25	0.20E-20	0.65E-16	0.19E-15	0.12E-11
20	0.0500	0.32E-29	0.16E-23	0.12E-18	0.16E-18	0.45E-14
21	0.0008	0.32E-26	0.25E-20	0.10E-15	0.33E-15	0.53E-11
22	0.0250	0.19E-27	0.18E-21	0.69E-17	0.48E-16	0.65E-12
23	5.0000	0.11E-23	0.69E-19	0.31E-15	0.12E-14	0.14E-11
24	0.0300	0.13E-28	0.60E-23	0.66E-18	0.89E-18	0.22E-13
25	0.0025	0.22E-29	0.10E-23	0.44E-19	0.45E-19	0.67E-15
26	0.0200	0.68E-24	0.17E-18	0.36E-14	0.91E-14	0.33E-10
27	0.0200	0.68E-24	0.17E-18	0.35E-14	0.91E-14	0.33E-10
28	0.0300	0.57E-27	0.55E-21	0.29E-16	0.75E-16	0.74E-12
29	0.0150	0.38E-28	0.39E-22	0.17E-17	0.31E-17	0.33E-13
30	0.0420	0.14E-30	0.23E-24	0.28E-19	0.50E-19	0.15E-14
31	0.0250	0.17E-29	0.31E-24	0.33E-19	0.16E-17	0.23E-13
32	100.0000	0.30E-29	0.41E-23	0.53E-18	0.74E-18	0.61E-13

Table 5: The 32 test problems solved using MIRK methods of orders 6,8,10 & 12. $E_{\max}$ (5.2) was used to measure the error. The problems were all solved with uniform meshes of 1025 points.

*This problem is of the form $DY = AY + b$, therefore MIRK 10 will have order 12 accuracy and Cash-Singhal 8 will have order 10.

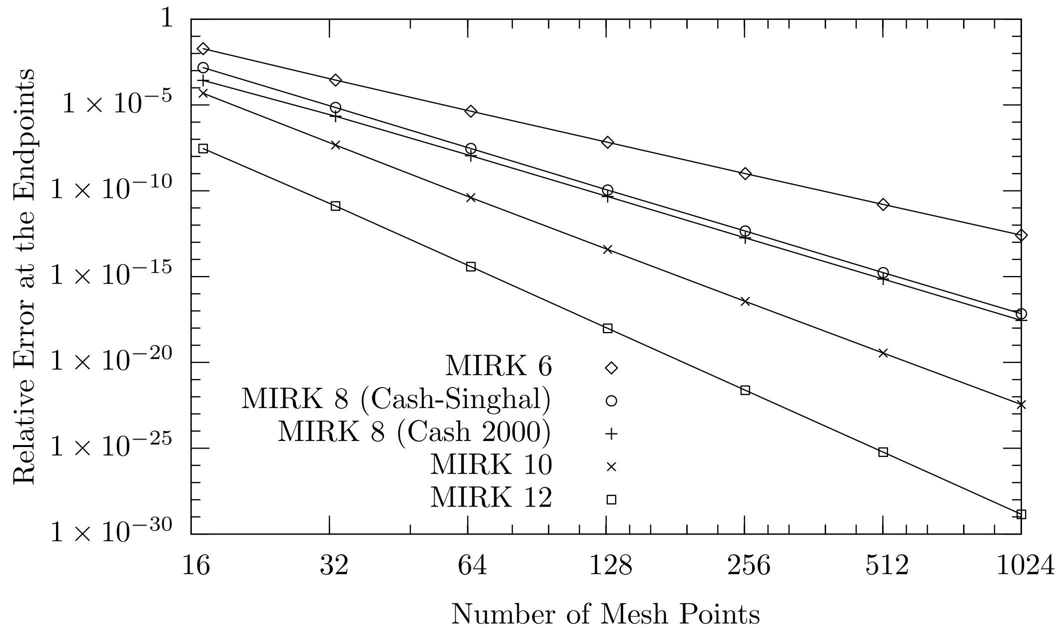

Figure 1: Log-log plot of the relative error in the endpoints vs. number of meshpoints for Orr-Sommerfeld 16 solved using MIRK 6 & 8 (Cash-Singhal) [7], MIRK 8 (Cash 2000) [3], MIRK 10 and MIRK 12.

D Coefficients for the MIRK10 Interpolant

$$\alpha_7 = \frac{\sqrt{297-132\sqrt{3}}}{66} \qquad \alpha_8 = \frac{\sqrt{297+132\sqrt{3}}}{66} \qquad \alpha_9 = \frac{\sqrt{33}}{22}$$

$$s_1^{\pm} = \frac{\left(3627+874\sqrt{3}\right)\left(7254-1748\sqrt{3}\mp 29927\,\alpha_7\right)}{43454004} \qquad s_2^{\pm} = \frac{\left(171+34\sqrt{3}\right)\left(213\,\alpha_7\pm 12\sqrt{3}\mp 52\right)}{618552}$$

$$s_3^{\pm} = \frac{\left(3340+9\sqrt{15}+362\sqrt{5}+1400\sqrt{3}\right)\left(1965\,\alpha_7\mp 56\sqrt{15}\alpha_5\mp 1360\,\alpha_5\mp 32\sqrt{5}\alpha_5\pm 240\sqrt{3}\alpha_5\right)}{19021200}$$

$$s_4^{\pm} = \frac{\left(3340+1400\sqrt{3}-362\sqrt{5}-9\sqrt{15}\right)\left(1965\,\alpha_7\mp 1360\,\alpha_6\pm 240\sqrt{3}\alpha_6\pm 32\sqrt{5}\alpha_6\pm 56\,\alpha_6\sqrt{15}\right)}{19021200}$$

$$t_1^{\pm} = \frac{\left(3627-874\sqrt{3}\right)\left(7254+1748\sqrt{3}\mp 29927\alpha_8\right)}{43454004} \qquad t_2^{\pm} = \frac{\left(171-34\sqrt{3}\right)\left(213\,\alpha_8\mp 52\mp 12\sqrt{3}\right)}{618552}$$

$$t_3^{\pm} = \frac{\left(3340-1400\sqrt{3}-362\sqrt{5}+9\sqrt{15}\right)\left(1965\,\alpha_8\mp 1360\,\alpha_5\pm 32\sqrt{5}\alpha_5\mp 240\sqrt{3}\alpha_5\mp 56\sqrt{15}\alpha_5\right)}{19021200}$$

$$t_4^{\pm} = \frac{\left(3340-1400\sqrt{3}+362\sqrt{5}-9\sqrt{15}\right)\left(1965\,\alpha_8\mp 240\sqrt{3}\alpha_6\mp 32\sqrt{5}\alpha_6\mp 1360\,\alpha_6\pm 56\,\alpha_6\sqrt{15}\right)}{19021200}$$

$$u_1^{\pm} = \frac{1}{2} \pm \frac{51}{2662}\sqrt{33} \qquad u_2^{\pm} = \mp\frac{83}{6655} - \frac{1}{1331}\sqrt{33}$$

$$u_3 = \frac{\left(5664+148\sqrt{15}\right)\alpha_5}{33275} \qquad u_4 = \frac{4}{33275}\sqrt{11}\left(122\sqrt{15}+207\right)\alpha_5$$

$$u_5^{\pm} = \frac{1}{945010}\sqrt{11}\left(34\sqrt{3}-171\right)\left(120\sqrt{3}+165\mp 142\,\alpha_8\right)$$

$$u_6^{\pm} = \frac{\left(377\sqrt{3}-378\right)\left(15\sqrt{33}-420\sqrt{11}\mp 1562\,\alpha_8\right)}{10395110}$$

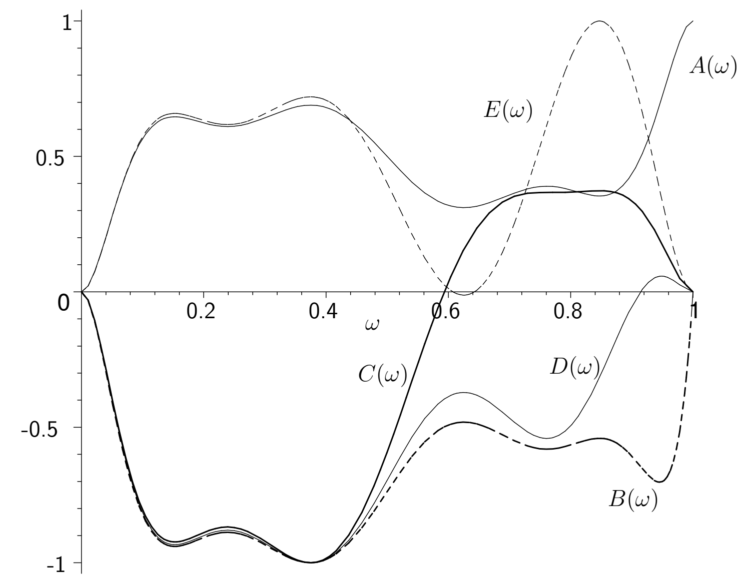

Figure 2: $A(\omega)$,...,$E(\omega)$ for our order 10 accurate Hermite-Birkhoff interpolant scaled up to pass through either ± 1

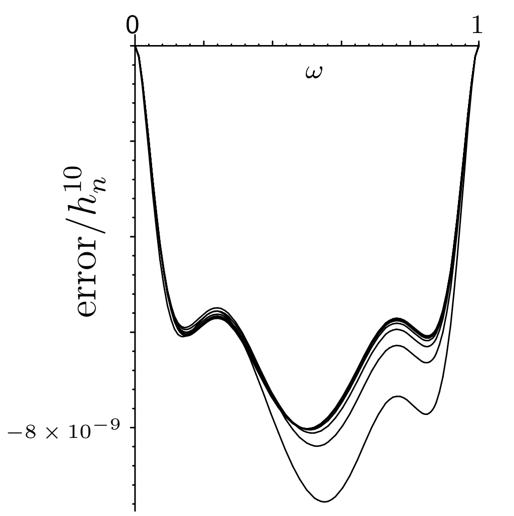

Figure 3: Verification of the order of accuracy of our Hermite-Birkhoff interpolant. For $h_n = 1, \frac{1}{2}, \ldots, \frac{1}{1024}$, $\left(y_{\text{exact}}(\omega) - y_{n+\omega}^{(10)}\right)/h_n^{10}$

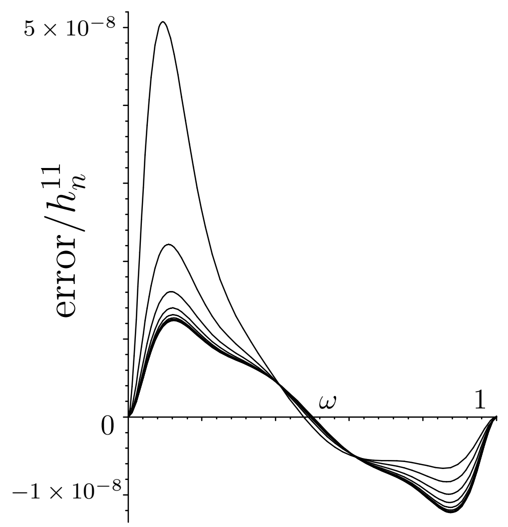

Figure 4: Confirmation that we computed the leading order error term of our Hermite-Birkhoff interpolant correctly. For $h_n = 1, \frac{1}{2}, \ldots, \frac{1}{1024}$, $\left(y_{\text{exact}}(\omega) - y_{n+\omega}^{(10)} + \epsilon_{10}(\omega)\right)/h_n^{11}$

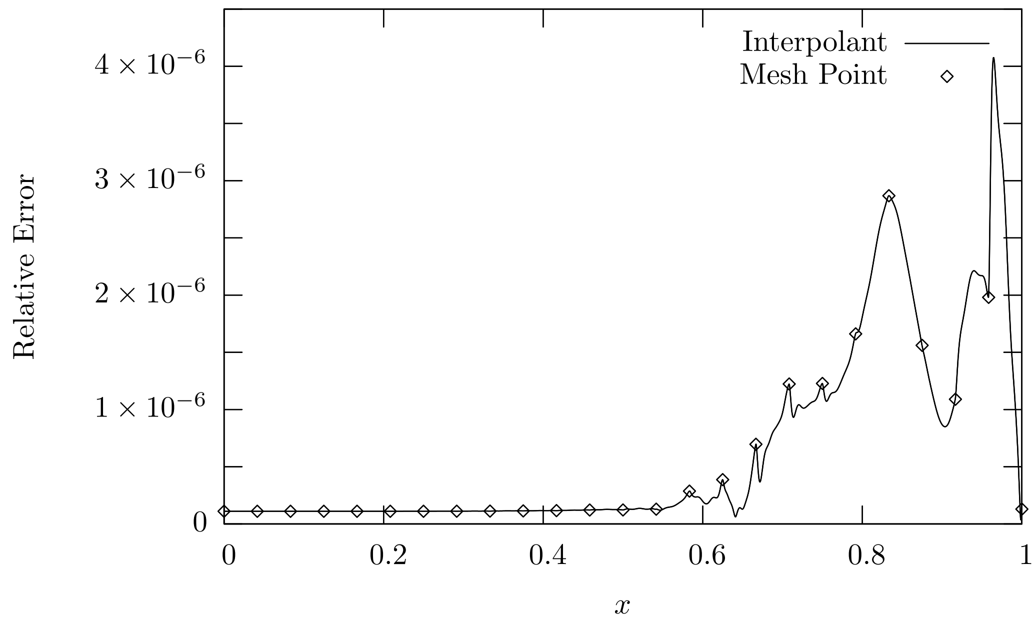

Figure 5: The error in the Orr-Sommerfeld 8 problem (4.1) solved using MIRK 10 with 24 piecewise interpolants

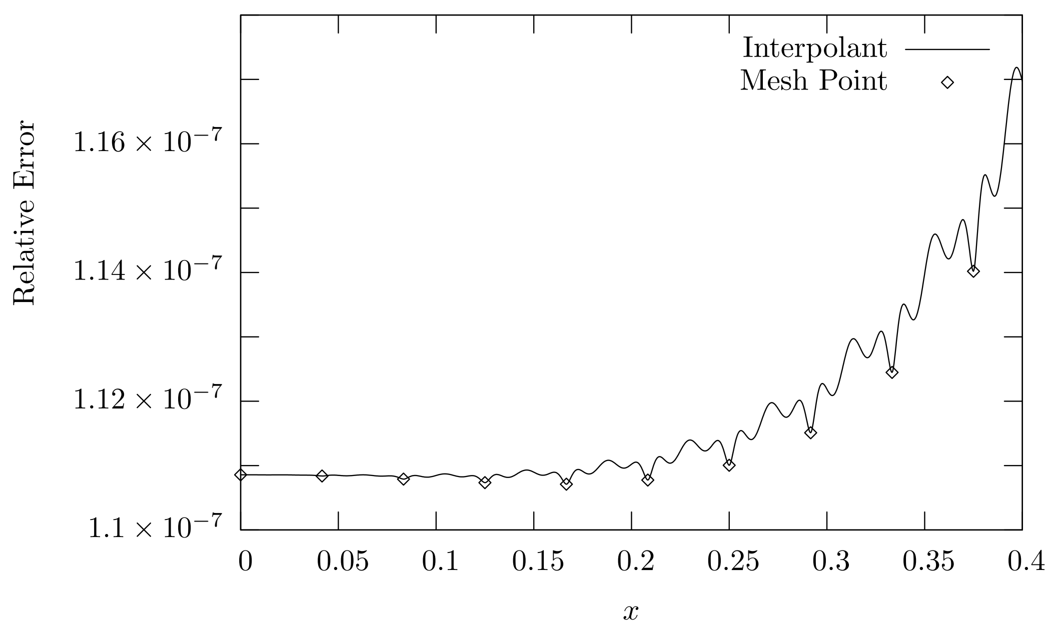

Figure 6: Enhancement of figure 5 localised to the interval $x \in [0, 0.4]$

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