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Fundamental Solution Method for Periodic Plane Elasticity[1]

H. Ogata[2]

Department of Computer Science,
Faculty of Electro-Communications,
The University of Electro-Communications,
1-5-1 Chofu-ga-Oka, Chofu, Tokyo 182-8585, Japan

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Abstract: We present a fundamental solution method for elasticity problems of planes with one-dimensional periodic structure, to which it is difficult to apply the conventional fundamental solution method. We propose an approximate solution of the fundamental solution method which is modified so that it is suitable to our problems. Numerical example is also included.

Keywords: fundamental solution method, periodic plane elasticity

Mathematics Subject Classification: 65N99

1 Introduction

Periodic elasticity problem is attractive subject and important from the theoretical and practical viewpoints. In this paper, we propose a fundamental solution method for elasticity problems of planes with one-dimensional periodic structure.

The fundamental solution method [4, 15] is a numerical solver for partial differential equation problems and is widely used in science and engineering, especially used for potential problems, where the method is called the "charge simulation method". This method approximates the solution by a superposition of the fundamental solutions of the differential operator and has the advantages that (i) it is easy to program, (ii) its computational cost is low and (iii) it achieves high accuracy such as exponential convergence under some conditions. However, the method had the weakness that it is difficult to apply the method to problems with spatial periodicity. The author overcame this weakness in the case of numerical conformal mappings [12] and in the case of Stokes flows [9, 10, 11].

In this paper, we present a fundamental solution method by modifying the ordinary fundamental solution method as in [13], whose approximate solution is given by a superposition of the displacement due to concentrated forces at discrete points, so that it is applicable to our problem with periodicity. We also show a numerical example of the presented method. The contents of this paper is as follows. In Section 2, we prepare some notations and formulate our problem. In Section

[1]Published electronically October 15, 2008
[2]Corresponding author. E-mail: ogata@im.uec.ac.jp

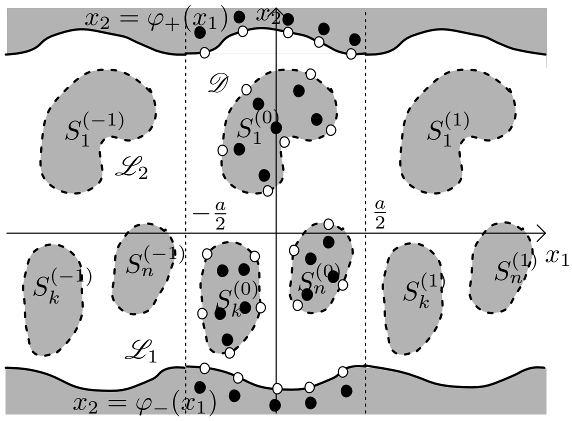


Figure 1: The strip plane $\mathscr{D}$ including infinite periodic arrays of holes $S_k^{(l)}$, $k = 1, 2, \ldots, n$, $l \in \mathbb{Z}$. The figure also includes the singularity points ζ_j ($\bullet$) and the collocation points z_i ($\circ$) (see Section 3). The figure also shows an example of the division of $\partial\mathscr{D}$ into $\mathscr{L}_1$ (solid lines) and $\mathscr{L}_2$ (broken lines).

3, we give a brief review of the ordinary fundamental solution method for two-dimensional plane elasticity given in [13] and, modifying this method, we present a fundamental solution method for periodic plane elasticity. In Section 4, we show the numerical examples of the presented method for strip planes with a periodic array of circular, elliptic and Cassini's oval holes. In addition, we computed the deformabilities of these planes by the presented method. In Section 5, we make the conclusion of this paper and refer to some problems for future studies related to this paper.

2 Formulation

In this section, we formulate our problem. Throughout this paper, we denote by $\mathbb{Z}$ the set of all the integers, by $\mathbb{R}$ that of all the real numbers and by $\mathbb{C}$ that of all the complex numbers. Besides, we equalise a point $\boldsymbol{x} = (x_1, x_2)$ in the two-dimensional Euclidean plane $\mathbb{R}^2$ with the complex number $z = x_1 + \mathrm{i}x_2$.

Let $a(> 0)$ be the period of our problem. Let the elastic plane $\mathscr{D}$ be between the periodic curves $x_2 = \varphi_+(x_1)$ and $x_2 = \varphi_-(x_1)$, where $\varphi_\pm(x_1)$ are periodic functions of period $a(> 0)$ such that $\varphi_+(x_1) > \varphi_-(x_1)$ ($\forall x_1 \in \mathbb{R}$), and possess an infinite periodic array of holes $S_k^{(l)}$, $k = 1, 2, \ldots, n,\ l \in \mathbb{Z}$ with boundaries consisting of smooth closed curves such that

$$S_k^{(m)} = S_k^{(l)} + (m - l)a = \{\, z + (m - l)a \mid z \in S_k^{(l)} \,\}, \quad k = 1, 2, \ldots, n; \quad l, m \in \mathbb{Z},$$
$$S_k^{(0)} \subset \left\{ z \in \mathbb{C} \;\middle|\; -\frac{a}{2} < \mathrm{Re}\, z \leq \frac{a}{2} \right\}, \quad k = 1, 2, \ldots, n$$

(see Fig.1).

Our problem is the boundary value problem of the elastostatic equation for the displacement vector $\boldsymbol{u}(\boldsymbol{x})$

$$(\lambda + G)\nabla(\nabla \cdot \boldsymbol{u}) + G\Delta\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathscr{D}, \tag{1}$$
$$\boldsymbol{u}(z) = \boldsymbol{u}_0(z) \quad \text{on } \mathscr{L}_1, \quad \boldsymbol{T}(z, \boldsymbol{n}) = \boldsymbol{T}_0(z) \quad \text{on } \mathscr{L}_2. \tag{2}$$

In (1), λ and G are Lamé's constants given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)} \tag{3}$$

with Young's modulus E and Poisson's ratio ν. In (2), $\mathscr{L}_1$ and $\mathscr{L}_2$ are parts of the boundary $\partial\mathscr{D}$ such that

$$\begin{gathered} \mathscr{L}_1 \cup \mathscr{L}_2 = \partial\mathscr{D} = \{\, z = x_1 + \mathrm{i}x_2 \in \mathbb{C} \mid x_2 = \varphi_\pm(x_1) \,\} \cup \bigcup_{l\in\mathbb{Z}} \bigcup_{k=1}^{N} S_k^{(l)}, \\ \mathscr{L}_1 \cap \mathscr{L}_2 = \emptyset \end{gathered} \tag{4}$$

(see Fig.1), $\boldsymbol{T}(z,\boldsymbol{n}) = [T_1(z,\boldsymbol{n}), T_2(z,\boldsymbol{n})]$ is the stress vector at a point z on a boundary with the unit normal vector $\boldsymbol{n} = [n_1, n_2]$, i.e. [3],

$$T_\alpha(z) = \sigma_{\alpha\beta}(z) n_\beta, \quad \alpha = 1,2 \tag{5}$$

and $\boldsymbol{u}_0(z) = [u_1^{(0)}(z), u_2^{(0)}(z)]$, $\boldsymbol{T}_0(z) = [T_1^{(0)}(z), T_2^{(0)}(z)]$ are given periodic functions of period a respectively defined on $\mathscr{L}_1$, $\mathscr{L}_2$.

3 Fundamental solution method

In this section, we present a periodic fundamental solution for our problems with spatial periodicity. For the discussion of our problem, we adopt the terminology based on complex analysis [6].

Let $\varphi(z)$ and $\psi(z)$ be the complex stress functions, that is, the analytic functions in $\mathscr{D}$ such that they give the Airy stress function $F(x_1,x_2)$, which is a biharmonic function, by

$$F(x_1,x_2) = \mathrm{Re}\left\{ \overline{z}\varphi(z) + \int^z \psi(z')\mathrm{d}z' \right\} \tag{6}$$

and gives the displacement vectors $\boldsymbol{u}(z) = [u_1(z), u_2(z)]$ and the stress tensor $[\sigma_{\alpha\beta}(z)]$ by

$$\begin{gathered} \sigma_{22} + \sigma_{11} = 4\,\mathrm{Re}\,\varphi'(z), \quad \frac{\sigma_{22}-\sigma_{11}}{2} + \mathrm{i}\sigma_{12} = \overline{z}\varphi''(z) + \psi'(z), \\ 2G(u_1 + \mathrm{i}u_2) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \end{gathered} \tag{7}$$

with $\kappa = 3 - 4\nu$. The ordinary fundamental solution method as in [13] approximates the solution by a linear combination of the displacements due to concentrated forces at discrete points [14]. In terms of the complex stress functions, it is expressed as

$$\varphi(z) \approx \frac{2}{\kappa}Q_0 - \sum_{j=1}^{N} Q_j \log(z-\zeta_j), \quad \psi(z) \approx \kappa\sum_{j=1}^{N} \overline{Q_j}\log(z-\zeta_j) + \sum_{j=1}^{N} \frac{\overline{\zeta_j}Q_j}{z-\zeta_j} \tag{8}$$

with singularity points ζ_j, $j = 1,2,\ldots,N$ given by the user outside the plane and complex coefficients $Q_j(= Q_1^{(j)} + \mathrm{i}Q_2^{(j)})$, $j = 0,1,2,\ldots,N$ which are to be determined so that the boundary conditions are approximately satisfied and are subject to the constraint

$$\sum_{j=1}^{N} Q_j = 0 \tag{9}$$

[3] We here apply the Einstein convention to suffices in Greek letters.

so that $\varphi(z)$ and $\psi(z)$ remain invariant with respect to affine transformations as the invariant scheme of the charge simulation method proposed by Murota [5]. The approximation (8) physically illustrates a superposition of the displacements due to the concentrated (fictitious) forces $\boldsymbol{F}_j = 2\pi(1+\kappa)[Q_1^{(j)}, Q_2^{(j)}]$ at the points ζ_j.

In applying the above approximation to our problem, it is natural to distribute the fundamental solution appearing in (8) in an infinite periodic array, namely,

$$\begin{aligned} \varphi(z) &\approx \frac{2}{\kappa}Q_0 - \sum_{m\in\mathbb{Z}}\sum_{j=1}^{N} Q_j \log(z-\zeta_j - ma), \\ \psi(z) &\approx \kappa \sum_{m\in\mathbb{Z}}\sum_{j=1}^{N} \overline{Q_j}\log(z-\zeta_j-ma) + \sum_{m\in\mathbb{Z}}\sum_{j=1}^{N}\frac{(\overline{\zeta_j}+ma)Q_j}{z-\zeta_j-ma}. \end{aligned} \tag{10}$$

However, the right-hand sides of the above approximations are not convergent. Therefore, we have to modify the approximations in (10) so that the infinite sums are convergent and the distribution of the singularities in (10) is not changed. Namely, we make the modifications[4]

$$\begin{aligned} &\sum_{m\in\mathbb{Z}}\sum_{j=1}^{N} Q_j \log(z-\zeta_j-ma) \\ &\to \sum_{j=1}^{N} Q_j \left\{ \log\frac{\pi}{a}(z-\zeta_j) + \sum_{m\neq 0}\left[\log\left(1-\frac{z-\zeta_j}{ma}\right)+\frac{z-\zeta_j}{ma}\right]\right\} = \sum_{j=1}^{N} Q_j \log\sin\left[\frac{\pi}{a}(z-\zeta_j)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} \sum_{m\in\mathbb{Z}}\sum_{j=1}^{N}\frac{(\overline{\zeta_j}+ma)Q_j}{z-\zeta_j-ma} &= \sum_{j=1}^{N} Q_j(z-2\mathrm{i}\,\mathrm{Im}\,\zeta_j)\sum_{m\in\mathbb{Z}}\frac{1}{z-\zeta_j-ma} \\ \to \sum_{j=1}^{N} Q_j(z-2\mathrm{i}\,\mathrm{Im}\,\zeta_j)&\left\{\frac{1}{z-\zeta_j}+\sum_{m\neq 0}\left(\frac{1}{z-\zeta_j-ma}+\frac{1}{ma}\right)\right\} \\ = \frac{\pi}{a}\sum_{j=1}^{N} Q_j(z-2\mathrm{i}\,\mathrm{Im}\,\zeta_j)&\cot\left[\frac{\pi}{a}(z-\zeta_j)\right], \end{aligned} \tag{12}$$

Then we obtain the approximation

$$\varphi(z) \approx \varphi_N(z) = \frac{2}{\kappa}Q_0 - \sum_{j=1}^{N} Q_j \log\sin\left[\frac{\pi}{a}(z-\zeta_j)\right], \tag{13}$$

$$\psi(z) \approx \psi_N(z) = \kappa\sum_{j=1}^{N}\overline{Q_j}\log\sin\left[\frac{\pi}{a}(z-\zeta_j)\right] + \frac{\pi}{a}\sum_{j=1}^{N} Q_j(z-2\mathrm{i}\,\mathrm{Im}\,\zeta_j)\cot\left[\frac{\pi}{a}(z-\zeta_j)\right] \tag{14}$$

with singularity points ζ_j, $j=1,2,\ldots,N$ given by the user outside the plane $\mathscr{D}$ (see Fig.1) and the complex coefficients $Q_j = Q_1^{(j)} + \mathrm{i}Q_2^{(j)}$, $j = 0,1,\ldots,N$ which are determined later and are subject to the constraint (9). The above approximation (13, 14) gives us the approximation of the

[4]We used the constraint (9) on the first equality of (12).

displacement vector $\boldsymbol{u}(z) = [u_1(z), u_2(z)]$

$$\begin{aligned} \boldsymbol{u}(z) \simeq \boldsymbol{u}_N(z) &= [u_1^{(N)}(z), u_2^{(N)}(z)], \\ u_\alpha^{(N)}(z) &= \frac{1}{G}\left\{ Q_\alpha^{(0)} + \sum_{j=1}^{N} \Gamma_{\alpha\beta}(z-\zeta_j) Q_\beta^{(j)} \right\}, \quad \alpha = 1,2 \end{aligned} \tag{15}$$

with

$$\begin{aligned} \left.\begin{matrix} \Gamma_{11}(z) \\ \Gamma_{22}(z) \end{matrix}\right\} &= -\kappa \log\left| \sin\frac{\pi z}{a} \right| \pm \frac{\pi}{a} \operatorname{Im} z \cdot \operatorname{Im}\cot\frac{\pi z}{a}, \\ \Gamma_{12}(z) = \Gamma_{21}(z) &= \frac{\pi}{a} \operatorname{Im} z \cdot \operatorname{Re}\cot\frac{\pi z}{a}. \end{aligned} \tag{16}$$

Additionally, the strain tensor $[\epsilon_{\alpha\beta}(z)]$ is approximated by

$$\epsilon_{\alpha\beta}(z) \simeq \epsilon_{\alpha\beta}^{(N)}(z) = \frac{1}{G} \sum_{j=1}^{N} e_{\alpha\beta\gamma}(z-\zeta_j) Q_\gamma^{(j)}, \quad \alpha,\beta = 1,2 \tag{17}$$

with

$$\begin{aligned} e_{111}(z) &= -\left\{ \kappa\frac{\pi}{a} \operatorname{Re}\cot\frac{\pi z}{a} + \left(\frac{\pi}{a}\right)^2 \operatorname{Im} z \cdot \operatorname{Im}\operatorname{cosec}^2\frac{\pi z}{a} \right\}, \\ e_{112}(z) &= -\left(\frac{\pi}{a}\right)^2 \operatorname{Re}\operatorname{cosec}^2\frac{\pi z}{a}, \\ e_{221}(z) &= \frac{\pi}{a} \operatorname{Re}\cot\frac{\pi z}{a} + \left(\frac{\pi}{a}\right)^2 \operatorname{Im} z \cdot \operatorname{Im}\operatorname{cosec}^2\frac{\pi z}{a}, \\ e_{222}(z) &= \frac{\pi}{a}(\kappa - 1) \operatorname{Im}\cot\frac{\pi z}{a} + \left(\frac{\pi}{a}\right)^2 \operatorname{Im} z \cdot \operatorname{Re}\operatorname{cosec}^2\frac{\pi z}{a}, \\ e_{121}(z) = e_{211}(z) &= \frac{\pi}{2a}(1+\kappa) \operatorname{Im}\cot\frac{\pi z}{a} - \left(\frac{\pi}{a}\right)^2 \operatorname{Im} z \cdot \operatorname{Re}\operatorname{cosec}^2\frac{\pi z}{a}, \\ e_{122}(z) = e_{212}(z) &= \frac{\pi}{2a}(1-\kappa) \operatorname{Re}\cot\frac{\pi z}{a} + \left(\frac{\pi}{a}\right)^2 \operatorname{Im} z \cdot \operatorname{Re}\operatorname{cosec}^2\frac{\pi z}{a}, \end{aligned} \tag{18}$$

and, by applying generalised Hooke's law to the approximate strain tensor $[\epsilon_{\alpha\beta}(z)]$, the stress tensor $[\sigma_{\alpha\beta}(z)]$ is approximated by

$$\sigma_{\alpha\beta}(z) \simeq \sigma_{\alpha\beta}^{(N)}(z) = \sum_{j=1}^{N} s_{\alpha\beta\gamma}(z-\zeta_j) Q_\gamma^{(j)}, \quad \alpha,\beta = 1,2 \tag{19}$$

with

$$s_{\alpha\beta\gamma}(z) = \lambda\delta_{\alpha\beta} e_{\mu\mu\gamma}(z) + 2G e_{\alpha\beta\gamma}(z), \quad \alpha,\beta,\gamma = 1,2. \tag{20}$$

The approximation (13,14) or (15) physically illustrates a superposition of the displacements due to an infinite periodic array of the concentrated forces $\boldsymbol{F}_j = 2\pi(1+\kappa)\boldsymbol{Q}_j = 2\pi(1+\kappa)[Q_1^{(j)}, Q_2^{(j)}]$ at the points $\zeta_j + ma$, $m \in \mathbb{Z}$. We remark that the approximate solution $\boldsymbol{u}_N(z)$ exactly satisfies the elasticity equation (1). As to the boundary conditions, we pose on the approximate solution the "collocation conditions", the conditions that the approximate solution collocationally satisfies the boundary conditions, i.e.,

$$\boldsymbol{u}_N(z_i) = \boldsymbol{u}_0(z_i),\ i = 1,2,\ldots,N', \quad \boldsymbol{T}_N(z_i, \boldsymbol{n}_i) = \boldsymbol{T}_0(z_i),\ i = N'+1, N'+2, \ldots, N \tag{21}$$

with the points $z_1, z_2, \ldots, z_{N'} \in \mathscr{L}_1$, $z_{N'+1}, z_{N'+2}, \ldots, z_N \in \mathscr{L}_2$, which are called the collocation points, given by the user (see Fig.1) and the unit normal vector $\boldsymbol{n}_i = [n_1^{(i)}, n_2^{(i)}]$, $i = N'+1, N'+2, \ldots, N$ at z_i, where $\boldsymbol{T}_N(z, \boldsymbol{n}) = [T_1^{(N)}(z, \boldsymbol{n}), T_2^{(N)}(z, \boldsymbol{n})]$ is the approximate stress vector at a point z and a unit normal vector $\boldsymbol{n} = [n_1, n_2]^t$ obtained from (19), i.e.,

$$T_\alpha(z, \boldsymbol{n}) = \sigma_{\alpha\beta}^{(N)}(z) n_\beta = \sum_{j=1}^{N} s_{\alpha\beta\gamma}(z - \zeta_j) n_\beta Q_\gamma^{(j)}, \quad \alpha = 1, 2. \tag{22}$$

The collocation conditions (21) are equivalent to the equations

$$\begin{aligned} Q_\alpha^{(0)} + \sum_{j=1}^{N} \Gamma_{\alpha\beta}(z_i - \zeta_j) Q_\beta^{(j)} &= G u_\alpha^{(0)}(z_i), \qquad i = 1, 2, \ldots, N'; \quad \alpha = 1, 2, \\ \sum_{j=1}^{N} s_{\alpha\beta\gamma}(z_i - \zeta_j) n_\beta^{(i)} Q_\gamma^{(j)} &= T_\alpha^{(0)}(z_i), \qquad i = N'+1, N'+2, \ldots, N; \quad \alpha = 1, 2, \end{aligned} \tag{23}$$

which and the constraint (9) form a linear system with respect to $\boldsymbol{Q}_j = [Q_1^{(j)}, Q_2^{(j)}]$, $j = 1, 2, \ldots, N$. We determine the coefficients Q_j by solving the linear system and obtain the approximate solution (13,14) or (15).

4 Numerical examples

In this section, we present some numerical examples. All the computation were carried out on a DELL Precision 670 workstation using programs coded in C++ with double precision working.

Example 1 The first example is the elasticity problem of the strip plane with an infinite array of circular holes

$$\mathscr{D}_1 = \{\, z \in \mathbb{C} \mid |\operatorname{Im} z| < d, \quad |z - na| > r \ (\forall n \in \mathbb{Z}) \,\} \quad (\, d = a \,) \tag{24}$$

with the boundary conditions that

- uniform pressure $P(> 0)$ per unit length is posed on the upper edge $\operatorname{Im} z = d$,
- the lower edge $\operatorname{Im} z = -d$ is fixed, and
- no tension is posed on the boundary of each hole,

or, equivalently,

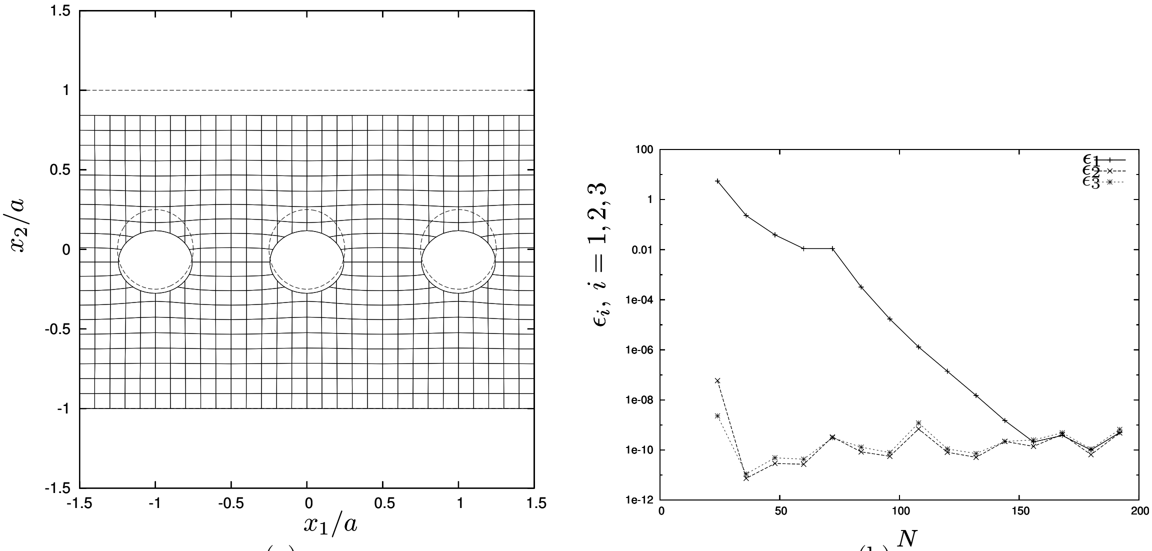
$$\begin{aligned} \boldsymbol{T}(z, \boldsymbol{n}) = [0, P] &\quad \text{if } \operatorname{Im} z = d, \\ \boldsymbol{u}(z) = \boldsymbol{0} &\quad \text{if } \operatorname{Im} z = -d, \\ \boldsymbol{T}(z, \boldsymbol{n}) = \boldsymbol{0} &\quad \text{if } |z - na| = r \text{ for some } n \in \mathbb{Z}. \end{aligned} \tag{25}$$

We solved the above problem by the presented method with the singularity points ζ_j and the collocation ones z_i given by

$$z_i = r \exp\left(\mathrm{i}\frac{2\pi(i-1)}{N_1}\right),\ \zeta_i = qr \exp\left(\mathrm{i}\frac{2\pi(i-1)}{N_1}\right), \qquad i = 1, 2, \ldots, N_1, \tag{26}$$

$$z_{i+N_1} = \frac{a(i-0.5)}{N_2} - \frac{a}{2} + \mathrm{i}d,\ \zeta_{i+N_1} = \frac{a(i-0.5)}{N_2} - \frac{a}{2} + \mathrm{i}2d, \qquad i = 1, 2, \ldots, N_2 \tag{27}$$

$$z_{i+N_1+N_2} = \frac{a(i-0.5)}{N_3} - \frac{a}{2} - \mathrm{i}d,\ \zeta_{i+N_1+N_2} = \frac{a(i-0.5)}{N_3} - \frac{a}{2} - \mathrm{i}2d, \quad i = 1, 2, \ldots, N_3, \tag{28}$$



(a) (b)

Figure 2: The elasticity of the plane with an infinite array of circular holes $\mathscr{D}_1$ with the radius of the holes $r = 0.25a$ computed by the presented method. The figure (a) shows the displacement of the rectangular meshes drawn on the plane $\mathscr{D}_1$ together with the boundaries of the plane without displacement in broken lines. The displacement shown in the figure (a) are enlarged by multiplying the actual displacement vectors by a scalar constant so that the displacement is visible. The figure (b) shows the error estimates ϵ_1, ϵ_2 and ϵ_3 on the boundaries of the plane $\mathscr{D}_1$ as functions of $N = N_1 + N_2 + N_3$.

where N_1, N_2, N_3 are positive integers such that $N = N_1 + N_2 + N_3$ gives the total number of the singularity/collocation points and q $(0 < q < 1)$ is an assignment parameter giving the position of the singularity points. Fig. 2 (a) shows the displacement of the rectangular meshes drawn on the plane $\mathscr{D}_1$. Fig. 3 shows the three-dimensional graphs of the elements of the strain tensor $\epsilon^{(N)}_{\alpha\beta}(z)$ and the stress tensors $\sigma^{(N)}_{\alpha\beta}(z)$. In the computations of Fig. 2, the radius of the circular holes r are taken as $r = 0.25a$, N_1, N_2, N_3 are taken as $N_1 = N_2 = N_3 = 32$ and q is taken as $q = 0.5$. To estimate the error of the presented method for this example, we computed the values

$$\epsilon_1 = \max\left\{ \frac{1}{P} \|\boldsymbol{T}_N(z, \boldsymbol{n}) - \boldsymbol{T}(z, \boldsymbol{n})| \;\middle|\; z \in \mathbb{C}, \quad |z| = r \right\}, \tag{29}$$

$$\epsilon_2 = \max\left\{ \frac{1}{P} \|\boldsymbol{T}_N(z, \boldsymbol{n}(z)) - \boldsymbol{T}(z, \boldsymbol{n}(z))\| \;\middle|\; z \in \mathbb{C},\ \mathrm{Im}\, z = d \right\}, \tag{30}$$

$$\epsilon_3 = \max\left\{ \frac{1}{a} \|\boldsymbol{u}_N(z) - \boldsymbol{u}(z)\| \;\middle|\; z \in \mathbb{C},\ \mathrm{Im}\, z = -d \right\}, \tag{31}$$

where the maxima are actually the maxima on 1000 boundary points distributed by the unit random numbers. Fig. 2 (b) shows the above error estimates as functions of the number of the points N with N_1, N_2, N_3 taken as $N_1 = N_2 = N_3 = N/3$. From the figure, we find that the error on the boundary of a hole ϵ_1 is dominant and decays exponentially as N increases. Then we computed the error estimate on the circular boundary ϵ_1 as a function of N_1 with N_2, N_3 fixed as

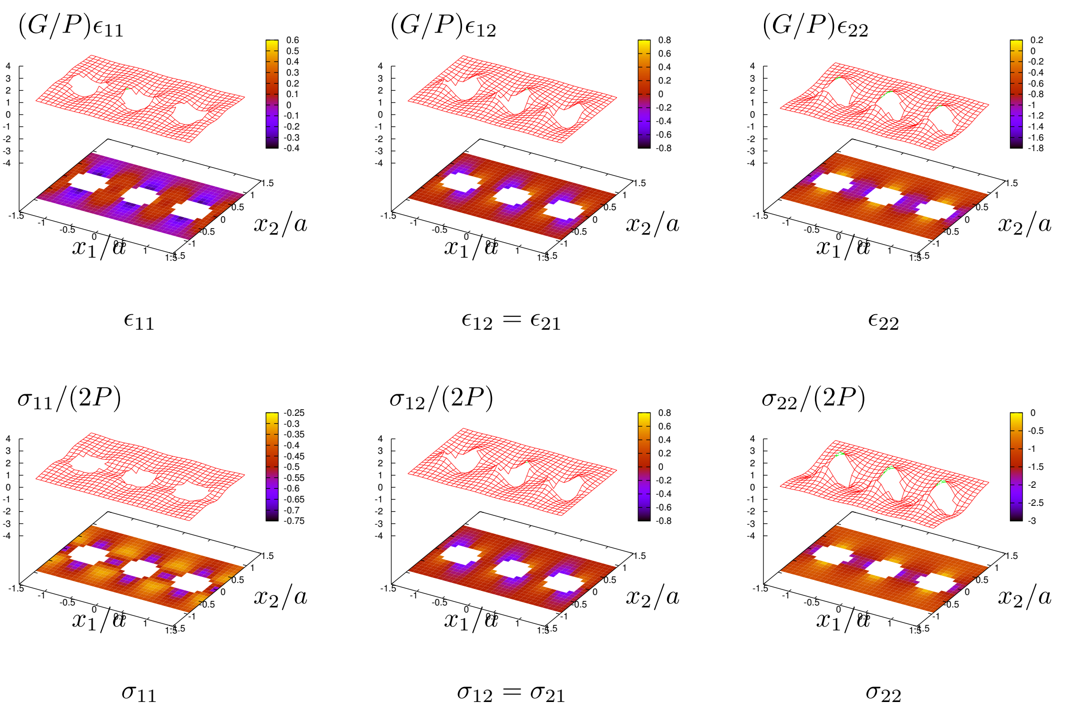


Figure 3: The three-dimensional graphs of the elements of the strain tensor $(G/P)\epsilon_{\alpha\beta}(z)$ and the stress tensor $\sigma_{\alpha\beta}(z)/(2P)$ on the elastic plane $\mathscr{D}_1$ with $r = 0.25a$.

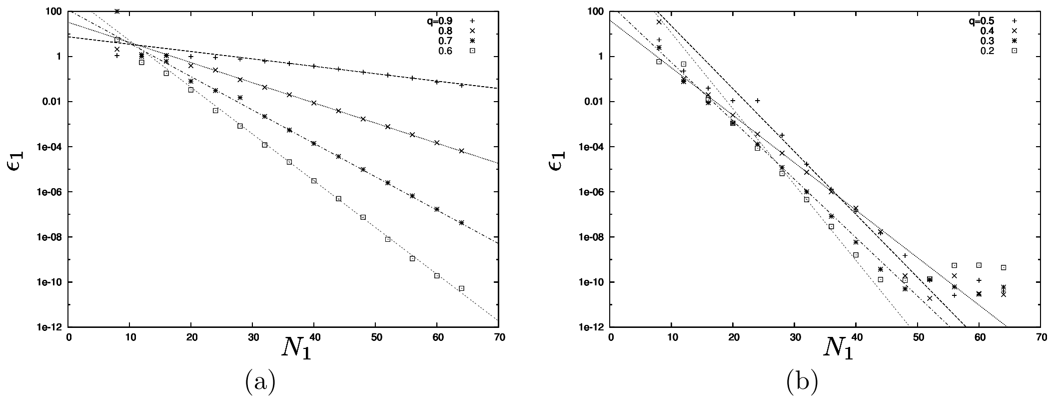


Figure 4: The values of the error estimate ε_1 for assignment parameter (a) $q = 0.9, 0.8, 0.7, 0.6$ and (b) $q = 0.5, 0.4, 0.3, 0.2$ as functions of N_1. In the figures, the lines of the formula $\varepsilon_1 = C \times \alpha^{N_1}$ are also included, where the constants C and α are obtained by the least square fittings.

Table 1: The values of α when we assume that the error estimate ϵ_1 obeys the form $\epsilon_1 = C \times \alpha^{N_1}$ with positive constants C, α.

q	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
α	0.93	0.81	0.71	0.62	0.53	0.66	0.55	0.46

$N_2 = N_3 = 12$ and, assuming that the error estimate ϵ_3 obeys the form

$$\varepsilon_1 \simeq C \times \alpha^{N_1} \tag{32}$$

with positive constant C and α independent of N_1, we computed the values of α by the least square fitting. Figure 4 and Table 1 show the results of the computations of the least square fittings. From Table 1, it seems that ϵ_1 obeys the formula

$$\epsilon_1 \simeq C \times q^{N_1} \tag{33}$$

if the assignment parameter q is near to 1.

We know no theoretical error estimate of the presented method but we here refer to a theoretical error estimate of the fundamental solution method applied to the Dirichlet problem of Laplace's equation on the two-dimensional domain exterior to a disk

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_1 \\ u = f & \text{on } \partial\Omega_1 \end{cases} \quad \text{with} \quad \Omega_1 = \{\, z \in \mathbb{C} \mid |z| > r \,\} \;\; (\, r > 0 \,). \tag{34}$$

As to the approximate solution of the fundamental solution method applied to the problem (34) [5]

$$u(z) \simeq u_N(z) = Q_j + \sum_{j=1}^{N} Q_j E(z - \zeta_j) \tag{35}$$

with the fundamental solution of the Laplace operator $E(z) = \dfrac{-1}{2\pi} \log |z|$ the positive coefficients Q_j, $j = 0, 1, \ldots, N$ determined by the collocation condition and the constraint $\sum_{j=1}^{N} Q_j = 0.$ and the singularity/collocation points given by (26), we have the following theorem.

Theorem 1 *Assume that the boundary data f in the problem (34) is real analytic, that is, the solution u has a harmonic extension on a domain $\{\, z \in \mathbb{C} \mid |z| > r_0 \,\}$ $(0 < r_0 < r)$. Then, as to the error of the approximate solution of the fundamental solution method (35) with the singularity points and the collocation points (26), we have the inequality*

$$\sup_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x}) - u_N(\boldsymbol{x})| \leqq C \left\{ \sup_{|\boldsymbol{x}| = r_0} |u(\boldsymbol{x})| \right\} \times \begin{cases} q^N & \text{if } q \geqq (r_0/r)^{1/2} \\ (r_0/r)^{N/2} & \text{if } q < (r_0/r)^{1/2} \end{cases} \tag{36}$$

for efficiently large N, where C is a positive constant independent of N and u.

This theorem can be proved in a way similar to the one of the theorems for the fundamental solution method applied to the Dirichlet problems of Laplace's equation [2]. The results of Theorem 1 coincide with the experimental result (33) for q sufficiently near to 1. It is obvious that Theorem 1

[5]This approximation is the "invariant" scheme of the fundamental solution method proposed by Murota [5].

given for Laplace's equation problems, that is, approximation problems of harmonic functions can not applied directly to our elasticity problems. However, taking into account that the presented method for our elasticity problems is based on the theory of biharmonic functions, there might be the possibility that we have a theoretical error estimate for our elasticity problem similar to Theorem 1.

As to the stability of the presented method, the linear system (9, 23) appearing in the presented fundamental solution method is so ill-conditioned that the condition number of the coefficient matrix of the linear system, which is estimated by Natori-Tsukamoto's method [7], is of order 10^{18} as $N_1 = N_2 = N_3 = 16$. It is, however, shown in [3] that the fundamental solution method applied to the Dirichlet problem of Laplace's equation gives an approximate solution with high accuracy though the linear system appearing in the method is ill-conditioned. Therefore, we guess that the presented method for our periodic elasticity problems also gives an approximate solution with high accuracy.

Example 2 The second example is the elasticity of the strip plane with an infinite array of elliptic holes

$$\mathscr{D}_2 = \left\{ z = x_1 + \mathrm{i}x_2 \in \mathbb{C} \;\middle|\; |\operatorname{Im} z| < d,\ \ \frac{(x_1 - na)^2}{A^2} + \frac{{x_2}^2}{B^2} > 1\ (\,\forall n \in \mathbb{Z}\,) \right\} \quad (\, d = a\,), \tag{37}$$

where A and B are constants such that $0 < A < d$, $0 < B < \min\{A, a/2\}$, with boundary conditions similar to the ones of Example 1, i.e.,

$$\begin{aligned} \boldsymbol{T}(z, n) &= [0, P] \quad \text{if } \operatorname{Im} z = d, \\ \boldsymbol{u}(z) &= \boldsymbol{0} \quad \text{if } \operatorname{Im} z = -d, \\ \boldsymbol{T}(z, n) &= \boldsymbol{0} \quad \text{if } \frac{(x_1 - na)^2}{A^2} + \frac{{y_1}^2}{B^2} = 1 \text{ for some } n \in \mathbb{Z}. \end{aligned} \tag{38}$$

We solved the above problem by the presented method with the singularity points ζ_j and the collocation ones z_i given by

$$z_i = J_c\left(r \exp\left(\mathrm{i}\frac{2\pi(i-1)}{N_1}\right)\right),\ \zeta_i = J_c\left(qr \exp\left(\mathrm{i}\frac{2\pi(i-1)}{N_0}\right)\right), \qquad i = 1, 2, \ldots, N_1, \tag{39}$$

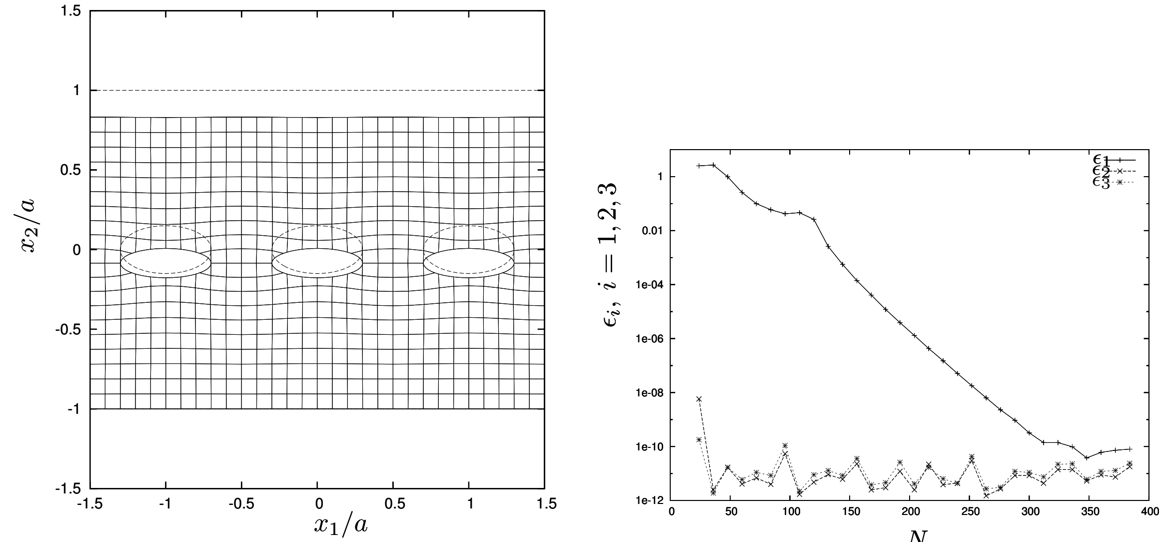
$$z_{i+N_1} = \frac{a(i-0.5)}{N_2} - \frac{a}{2} + \mathrm{i}d,\ \zeta_{i+N_1} = \frac{a(i-0.5)}{N_2} - \frac{a}{2} + \mathrm{i}2d, \qquad i = 1, 2, \ldots, N_2, \tag{40}$$

$$z_{i+N_1+N_2} = \frac{a(i-0.5)}{N_3} - \frac{a}{2} - \mathrm{i}d,\ \zeta_{i+N_1+N_2} = \frac{a(i-0.5)}{N_3} - \frac{a}{2} - \mathrm{i}2d, \qquad i = 1, 2, \ldots, N_3, \tag{41}$$

where $z = J_c(w)$ is the Joukowski transformation

$$z = J_c(w) = \frac{c}{2}\left(w + \frac{1}{w}\right), \quad c = \sqrt{A^2 - B^2}, \tag{42}$$

$r = \sqrt{(A+B)/(A-B)}$, q ($\rho^{-1} < q < 1$) is an assignment parameter giving the position of the singularity points and N_1, N_2, N_3 are positive integers such that $N = N_1 + N_2 + N_3$ gives the total number of the singularity/collocation points. Fig. 5 (a) shows the displacement of the rectangular meshes drawn on the plane $\mathscr{D}_2$. Fig. 6 shows the three-dimensional graphs of the elements of the strain tensor $\epsilon_{\alpha\beta}$ and the stress tensor $\sigma_{\alpha\beta}$. In Fig. 5 (a) and Fig. 6, the constants A, B are taken respectively as $A = 0.3a, B = 0.15a$, q is taken as $q = 0.6$ and N_1, N_2, N_3 are taken as $N_1 = N_2 = N_3 = 32$.



(a) (b)

Figure 5: The elasticity of the plane with an infinite periodic array of elliptic holes $\mathscr{D}_2$ with $A = 0.3a$ and $B = 0.15a$ computed by the presented method. The figure (a) shows the displacement of the rectangular meshes drawn on the plane $\mathscr{D}_2$ together with the boundaries of the plane without displacement in broken lines. The displacement shown in the figure (a) are enlarged by multiplying the actual displacement vectors by a scalar constant so that the displacement is visible. The figure (b) shows the error estimates on the boundaries of the plane $\mathscr{D}_2$.

Fig. 5 (b) shows the error estimates ϵ_2, ϵ_3 on the upper and lower edges respectively given by (30), (31) and ϵ_1 given by

$$\epsilon_1 = \max \left\{ \left. \frac{1}{P} \| \boldsymbol{T}_N(z, \boldsymbol{n}) - \boldsymbol{T}(z, \boldsymbol{n}) \| \ \right| \ z = x_1 + \mathrm{i} x_2 \in \mathbb{C}, \quad \frac{{x_1}^2}{A^2} + \frac{{x_2}^2}{B^2} = 1 \right\} \tag{43}$$

for the elasticity problem of the plane $\mathscr{D}_2$ with $A = 0.3a$ and $B = 0.15a$ as functions of the total number of the singularity/collocation points $N = N_1 + N_2 + N_3$ with N_1, N_2, N_3 taken as $N_1 = N_2 = N_3$. The maxima giving $\epsilon_1, \epsilon_2, \epsilon_3$ in (43), (30), (31) are, as in Example 1, the maxima on 1000 points distributed on the boundaries by the uniform random numbers. Fig. 5 (b) shows the error estimates ϵ_1, ϵ_2 and ϵ_3 as functions of the total number of the singularity/collocation points $N = N_1 + N_2 + N_3$. From the figure, we find that the error on the boundary of an elliptic hole ϵ_1 is dominant but the decay of ϵ_1 is slower than the one in Example 1.

Also in this example, it seems that the error on the holes ϵ_1 decays exponentially as a function of N.

Then we computed the error estimate on the elliptic holes ϵ_1 as a function of N_1 with N_2, N_3 fixed as $N_2 = N_3 = 16$ and, assuming that ϵ_1 obeys the form

$$\epsilon_1 = C \times \alpha^{N_1} \tag{44}$$

with constants C, α such that $C > 0$, $0 < \alpha < 1$, we computed the values of α by the least square fittings. Fig 7 and Table 2 show the results of the computations of α. From Table 2, it seems that we have

$$\epsilon_1 \simeq C \times q^{N_1} \tag{45}$$

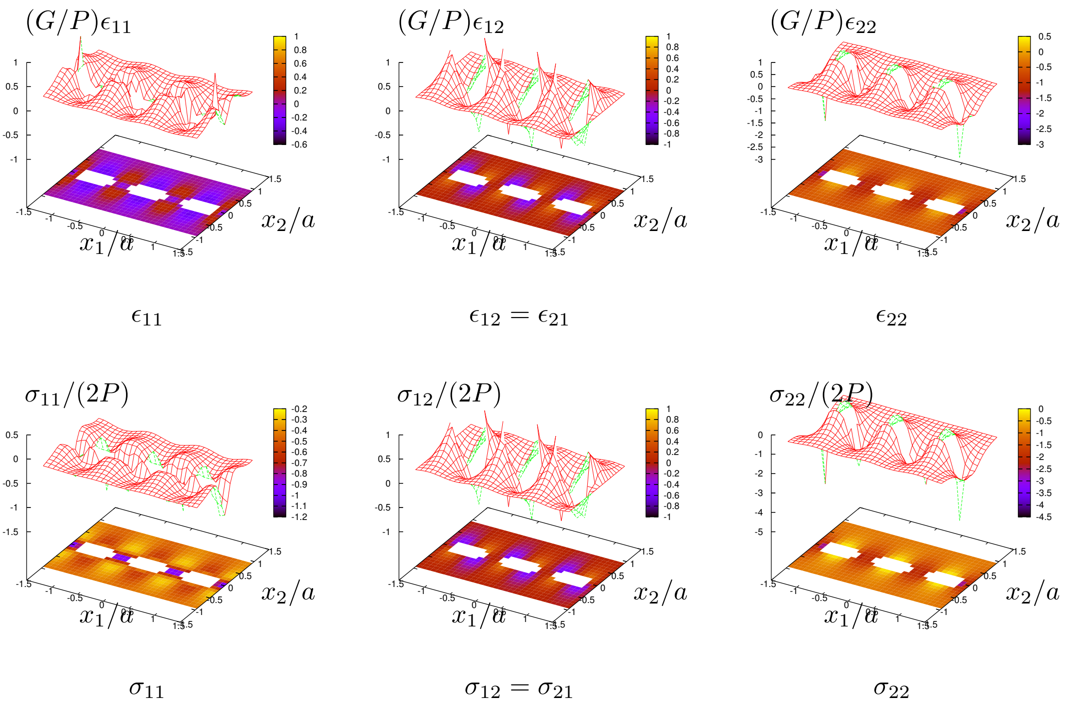


Figure 6: The three-dimensional graphs of the elements of the strain tensor $(G/P)\epsilon_{\alpha\beta}$ and the stress tensor $\sigma_{\alpha\beta}/(2P)$ on the elastic plane $\mathscr{D}_2$.

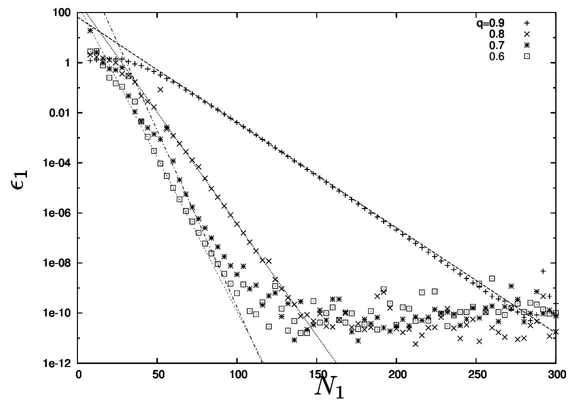


Figure 7: The error estimates on the holes ϵ_1 as functions of N_1 with $q = 0.9, 0.8, 0.7, 0.6$ and N_2, N_3 taken as $N_2 = N_3 = 16$. The figure also includes the lines of the formula $\epsilon_1 = C \times \alpha^{N_1}$ with C, α obtained by the least square fittings for each q.

with a positive constant C independent of N_1 as the assignment parameter q is sufficiently near to 1.

Also in this example, we know no theoretical error estimate of the presented method but we here refer to the following theorem for theoretical error estimate of the fundamental solution method applied to a Dirichlet problem of Laplace's equation on the domain exterior to an ellipse

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_2 \\ u = f & \text{on } \partial\Omega_2 \end{cases} \quad \text{with} \quad \Omega_2 = \left\{ z = x_1 + \mathrm{i}x_2 \in \mathbb{C} \;\middle|\; \frac{{x_1}^2}{A^2} + \frac{{x_2}^2}{B^2} > 1 \right\} \quad (\,0 < B < A\,), \tag{46}$$

namely, the approximate solution (35) with the singularity/collocation points (39).

Theorem 2 *Assume that the boundary data f in the problem (46) is real analytic, that is, the solution u has a harmonic extension on a domain $\{ J_c(\zeta) \mid \zeta \in \mathbb{C},\ |\zeta| > r_0 \}$ $(\,0 < r_0 < r\,)$, where $J_c(\cdot)$ is the Joukowski transformation (42). Then, for the error of the approximate solution of the fundamental solution method (35) applied to the Dirichlet problem of Laplace's equation on the domain exterior to an ellipse (46) with the singularity/collocation points (39), we have the inequality*

$$\sup_{z\in\Omega_2} |u(z) - u_N(z)| \leqq C\left\{ \sup_{|\zeta|=r_0} |u(J_c(\zeta))| \right\} q^N \tag{47}$$

if q is sufficiently near to 1, where C is a positive constant independent of N and u.

This theorem can be proved in a way similar to the one of the theorem for the fundamental solution method applied to Laplace's equation problem on an elliptic domain [8]. The inequality (47) coincides with the experimental result (45) for our problems. It is obvious that we cannot apply Theorem 2 to the presented method for Example2. However, as mentioned in Example 1, there might be the possibility that we have a theoretical error estimate for Example 2 similar to the one of Theorem 2.

As to the stability of the presented method to Example 2, the linear system appearing in the method is also so ill-conditioned that the condition number of the coefficient matrix of the linear system is of order 10^{19} as $N_1 = N_2 = N_3 = 16$. However, as discussed in Example 1, we guess that the presented method gives an approximate solution with high accuracy though the linear system (9, 23) is ill-conditioned.

Table 2: The value of α when we assume that the error estimate on the holes ϵ_1 obeys the formula (44).

q	0.9	0.8	0.7	0.6
α	0.91	0.78	0.72	0.75

Example 3 The last example is the elasticity of the strip plane with an infinite periodic array of Cassini's oval holes

$$\mathscr{D}_3 = \left\{ \, z \in \mathbb{C} \; \middle| \; |\operatorname{Im} z| < d, \; |(z - na)^2 - \alpha^2| > \beta^2 \, \right\} \quad (\, d = a \,) \tag{48}$$

where α,β are positive constants such that $\alpha < \beta < \sqrt{2}\alpha$ and taken here as $\alpha = 0.25a$, $\beta = 2^{1/8}\alpha$, with boundary conditions similar to the ones of Example 1 and Example 2. We solved the above problem by the presented method with the collocation points z_i given by

$$z_i = r_i \mathrm{e}^{\mathrm{i}\theta_i} \quad \text{with} \quad \theta_i = \frac{2\pi(i-1)}{N_1}, \quad r_i = \left\{ \alpha^2 \cos 2\theta_i + (\beta^4 - \alpha^4 \sin^2 2\theta_i)^{1/2} \right\}^{1/2}, \quad i = 1, 2, \ldots, N_1, \tag{49}$$

$$z_{i+N_1} = \frac{a(i-0.5)}{N_2} + \frac{a}{2} + \mathrm{i}d, \quad i = 1, 2, \ldots, N_2, \tag{50}$$

$$z_{i+N_1+N_2} = \frac{a(i-0.5)}{N_3} + \frac{a}{2} - \mathrm{i}d, \quad i = 1, 2, \ldots, N_3 \tag{51}$$

and the singularity points ζ_j given by

$$\zeta_j = z_j + \mathrm{i}q \times \begin{cases} (z_{j+1} - z_{j-1}), & j = 2, 3, \ldots, N_1 - 1 \\ (z_2 - z_{N_1}), & j = 1 \\ (z_1 - z_{N_1-1}), & j = N_1, \end{cases} \tag{52}$$

$$\zeta_{j+N_1} = \frac{a(j-0.5)}{N_2} + \frac{a}{2} + \mathrm{i}2d, \quad j = 1, 2, \ldots, N_2, \tag{53}$$

$$\zeta_{j+N_1+N_2} = \frac{a(j-0.5)}{N_3} + \frac{a}{2} - \mathrm{i}2d, \quad j = 1, 2, \ldots, N_3, \tag{54}$$

where N_1, N_2, N_3 are positive integers and $q(> 0)$ is an assignment parameter giving the position of the singularity points. Fig. 8 shows the displacement of the rectangular meshes drawn on the plane $\mathscr{D}_3$. Fig. 10 shows the three-dimensional graphs of the elements of the strain tensor $\epsilon_{\alpha\beta}$ and the stress tensor $\sigma_{\alpha\beta}$. In the computations of Fig. 8 and Fig. 10, N_1, N_2, N_3 are taken as $N_1 = N_2 = N_3$ and q is taken as $q = 4$.

Fig. 9 shows the error estimates ϵ_2, ϵ_3 on the upper and lower edges respectively given by (30), (31) and ϵ_1 on the boundary of a Cassini's oval hole given by

$$\epsilon_1 = \max \left\{ \, \frac{1}{P} \| \boldsymbol{T}_N(z, \boldsymbol{n}) - \boldsymbol{T}(z, \boldsymbol{n}) \| \; \middle| \; z \in \mathbb{C}, \quad |z^2 - \alpha^2| = \beta^2 \, \right\} \tag{55}$$

of the presented method with assignment parameter $q = 4, 2, 1, 0.5$ applied to the elasticity problem of the plane $\mathscr{D}_3$ with $\alpha = 0.25a$ and $\beta = 2^{1/8}\alpha$ as functions of the number of the points N. In the computations of Fig. 9, N_1, N_2, N_3 are taken as $N_1 = N_2 = N_3 = N/3$ and, as in Examples 1 and 2, the maxima giving the error estimates ϵ_1, ϵ_2, ϵ_3 in (55), (30), (31) are actually the maxima

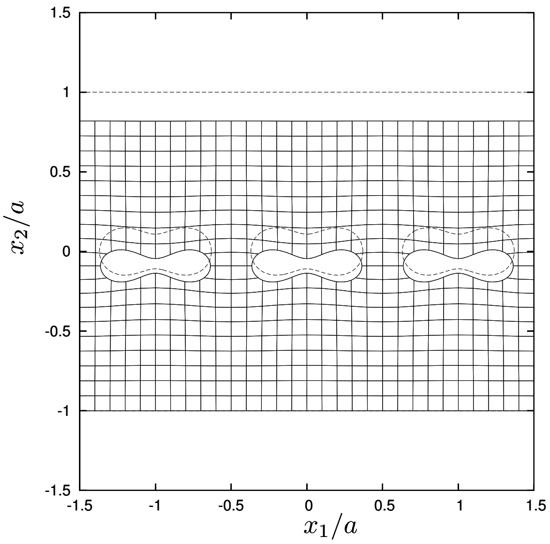


Figure 8: The elasticity of the plane with an infinite periodic array of Cassini's oval holes $\mathscr{D}_3$ with $\alpha = 0.25a$ and $\beta = 2^{1/8}\alpha$ computed by the presented method. The figure shows the displacement of the rectangular meshes drawn on the plane $\mathscr{D}_3$ together with the boundaries of the plane without displacement in broken lines. The displacement shown in the figure are enlarged by multiplying the actual displacement vectors by a scalar constant so that the displacement is visible.

on 1000 points on the boundaries distributed by the uniform random numbers. In all the cases of Fig. 9, we find that the error on the boundary of a hole ϵ_1 is dominant. In the cases of $q = 4, 2$, the error estimate ϵ_1 first becomes large and then decays exponentially as N increase. In the cases of $q = 1, 0.5$, the error estimate ϵ_1 does not decay much. As far as in the computations of Fig. 9, the choice of assignment parameter $q = 2$ gives the most accurate solution and the most stable performance.

As to the stability of the presented method applied to Example 3, the linear system (23,9) is so ill-conditioned that the condition number of the coefficient matrix of the linear system is of order 10^{17} as $N_1 = N_2 = N_3 = 16$. However, as discussed in Examples 1, 2, we guess that the presented method gives an accurate solution by choosing properly the assignment parameter q though the linear system (9, 23) is ill-conditioned.

Deformability of Planes We investigated how deformable the planes with an infinite periodic array of holes of the above three examples become as the holes becomes large. Namely, we computed by the presented method the displacements of the upper edges Δx_2 as functions of

$$\text{the area concentration} = \frac{\text{the area of a hole}}{2a}, \tag{56}$$

which illustrates how much the area is occupied by a hole per period. The results are shown in Fig. 11. From the figure, we find that the deformabilities are in the order: a plane with (1) elliptic holes, (2) circular holes and (3) Cassini's oval holes. In other words, a plane with Cassini's oval holes is the most rigid against pressure.

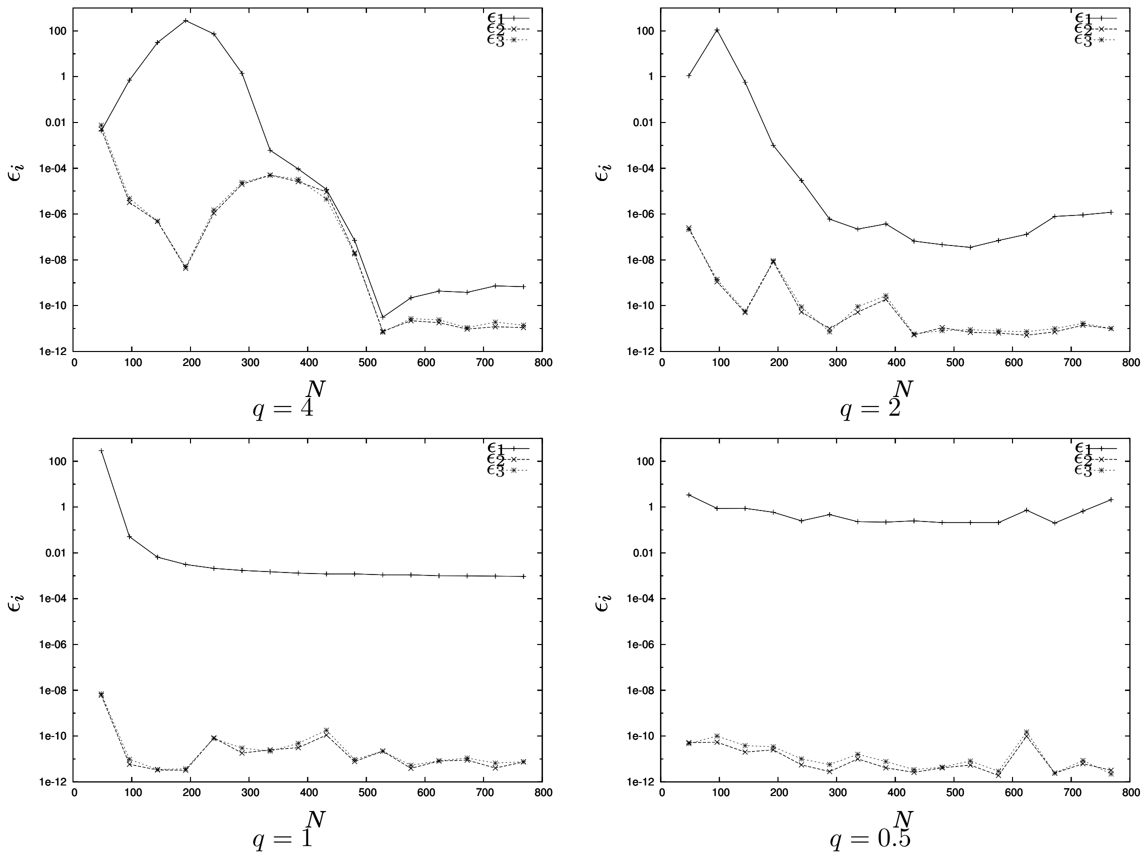


Figure 9: The error estimates ϵ_i, $i = 1, 2, 3$ of the presented method applied to Example 3 with assignment parameter $q = 4, 2, 1, 0.5$ as functions of the total number of the singularity/collocation points N. In the computations, the integers N_1, N_2, N_3 are taken as $N_1 = N_2 = N_3 = N/3$.

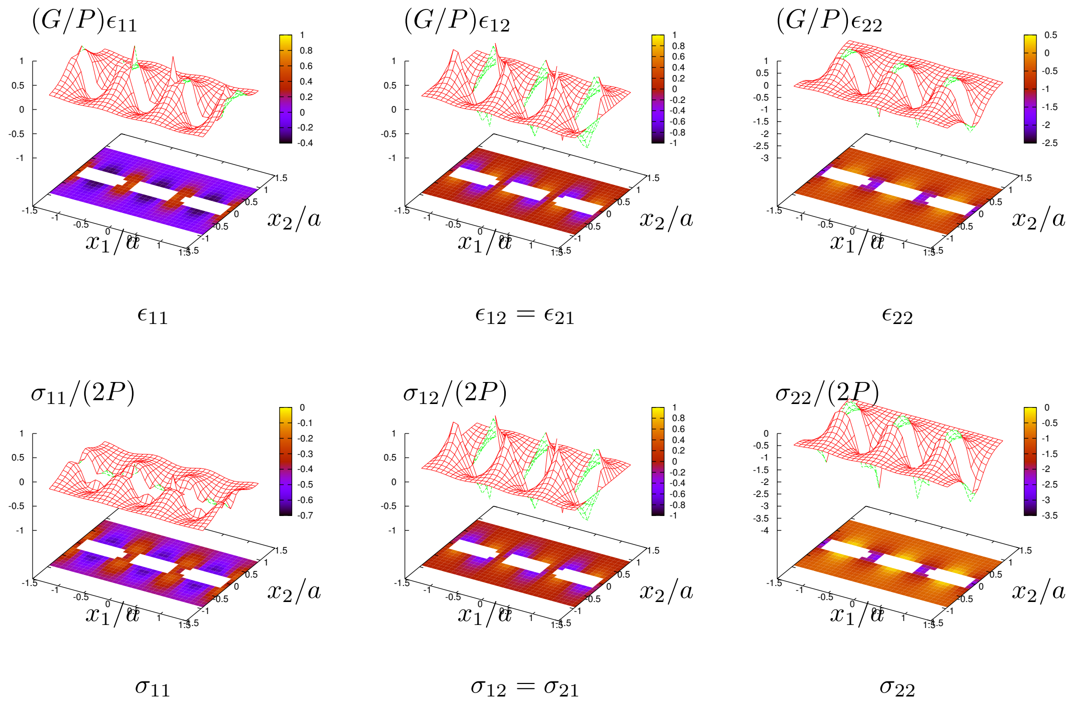


Figure 10: The three-dimensional graphs of the elements of the strain tensor $(G/P)\epsilon_{\alpha\beta}$ and the stress tensor $\sigma_{\alpha\beta}/(2P)$ on the elastic plane $\mathscr{D}_3$.

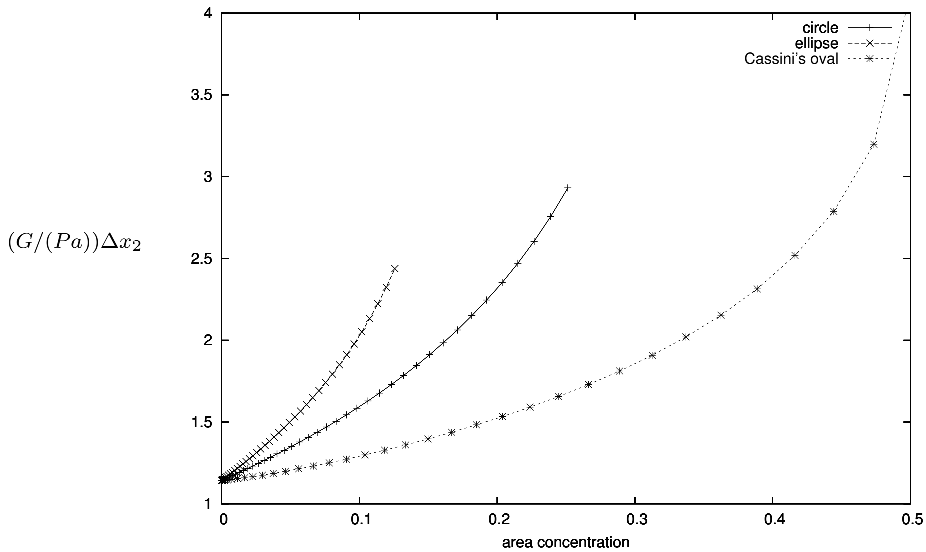


Figure 11: The displacement of the upper edge $(G/(Pa))\Delta x_2$ of the planes $\mathscr{D}_1$ (with circular holes), $\mathscr{D}_2$ (with elliptic holes) and $\mathscr{D}_3$ (with Cassini's oval holes).

5 Concluding Remarks

In this paper, we proposed a fundamental solution method for elasticity problems of planes with one-dimensional infinite periodic structure. The presented method is based on the two-dimensional elasticity theory described in terms of complex analysis and gives an approximate solution by a linear combination of the fundamental solutions of the elasticity equation with an infinite periodic array of sources instead of the fundamental solution with an isolated source. Numerical examples are shown for the elasticity problems of strip planes with an infinite periodic array of holes of three types, namely, circular, elliptic and Cassini's oval holes. They include the computations of the displacements, the strain tensors and the stress tensors on the planes and error estimates of the method, which show a sufficient accuracy of the method. The computations of the deformabilities of the planes are also presented and show that plane with Cassini's oval holes is the most rigid among planes of the three types.

Problems for future studies related to this paper are as follows.

- Extensions of the presented method to three-dimensional cases. In this extensions, the elasticity theory based on complex analysis will not be applicable and, instead, techniques based on the Fourier series as the ones in the periodic fundamental solution of the Stokes flow equation by Hasimoto [1].
- Applications of the presented method in science and engineering, for example, the analysis of the elasticity of porous media and the design of porous materials which are rigid against pressure, etc.

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